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ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." An explanation is provided of the languages of strings and arrows or classification and relations. This guide emphasizes functions as well. The String Game is presented as a means of using sets to stimulate logical thinking. Geometry, Measurement, Probability, and Statistics comprise the latter half of the guide. (Author/SA)

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Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version
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CSMP Mathematics for the Intermediate Grades
Part II

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics

U.S. DEPARTMENT OF HEALTH,
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Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades

Part II

The Languages of Strings and Arrows

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR L – LESSONS

WEEK	Languages of Strings and Arrows	Materials
1	**1L** L1.1: Composition of Relations p. 5	Teacher: Colored chalk; IG-II Language Poster #1 Student: Colored pencils; Worksheets L01, L02, L03, L04
2	**2L** L2.1: The String Game p. 72	T: The String Game S: None
3	**3L** L2.2: String Game Analysis #1 p. 75	T: The String Game; 2 Demonstration Analysis sheets S: String Game Analysis Pad
4	ADJUSTMENT DAY	
5	**4L** L1.2: Functions p. 23	T: Colored chalk S: Colored pencils; paper; Worksheets L05, L06
6	**5L** L2.3: Logical Thinking #1 p. 86	T: Colored chalk S: Colored pencils; paper
7	**6L** L1.3: With and Against p. 35	T: Colored chalk S: Colored pencils; Worksheets L07, L08, L09, L10
8	**7L** L2.4: String Game Analysis #2 p. 97	T: The String Game; 2 Demonstration Analysis Sheets S: String Game Analysis Pad; Worksheets L17, L18, L19, L20
9	**8L** L2.5: String Game Analysis #3 p. 103	T: The String Game; 4 Demonstration Analysis Sheets S: String Game Analysis Pad

SUGGESTED SCHEDULE FOR L–LESSONS

WEEK	Languages of Strings and Arrows	Materials
10	ADJUSTMENT DAY	
11	[redacted]	[redacted]
12	[redacted]	[redacted]
13	**9 L** L1.4: Arrow Messages #1 p. 46	T: Colored chalk S: Colored pencils; paper
14	**10L** L2.6: Logical Thinking #2 p. 107	T: The String Game; colored chalk S: None
15	**11L** L2.7: String Game Analysis #4 p. 115	T: The String Game; 4 Demonstration Analysis Sheets S: String Game Analysis Pad
16	**12L** L1.5: Arrow Messages #2 p. 61	T: Colored chalk S: Colored pencils; Worksheets L11, L12, L13, L14, L15, L16
17	**13L** L2.8: String Game Analysis #5 p. 119	T: The String Game S: String Game Analysis Pad Worksheets L21, L22
18	ADJUSTMENT DAY	

SUGGESTED CHRONOLOGICAL ORDER OF L-LESSONS

Lesson One L1.1 - Composition of Relations 5
Lesson Two L2.1 - The String Game 72
Lesson Three L2.2 - String Game Analysis #1 75
Lesson Four L1.2 - Functions 23
Lesson Five L2.3 - Logical Thinking #1 86
Lesson Six L1.3 - With and Against 35
Lesson Seven L2.4 - String Game Analysis #2 97
Lesson Eight L2.5 - String Game Analysis #3 103
Lesson Nine L1.4 - Arrow Messages #1 46
Lesson Ten L2.6 - Logical Thinking #2 107
Lesson Eleven L2.7 - String Game Analysis #4 115
Lesson Twelve L1.5 - Arrow Messages #2 61
Lesson Thirteen L2.8 - String Game Analysis #5 119

THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings represents, at a nonverbal level, the language of sets, that is, the grouping together of objects whether the objects are numbers or not. The language of arrows, on the other hand, is the nonverbal language of relations among objects of the same or different sets. Each of these languages permeates the different content strands. They are sometimes used separately, sometimes together to present an unending variety of numerical and nonnumerical situations to the student.

Classification: The Language of Strings

Classifying things is one of the most important and all-pervasive of human intellectual activities. It is one of the fundamental tools we use to get a handle on things--to keep them straight in our minds, and to avoid being overwhelmed by a sea of chaos. We classify all the time in our everyday life--often unconsciously or, at least, without much deliberate thought: we classify cars by brand (Ford, Chevrolet, etc.); we classify people by citizenship (American, English, German, Australian); we classify the letters of the alphabet into vowels and consonants, and so on. On a more systematic level, much of science is the search for apt and useful systems of classification: the astronomer sorts out the heavenly bodies into stars, planets, moons, comets; the zoologist sorts out animals into reptiles, birds, mammals, insects, protozoa, etc.; the botanist does the same with plants; the linguist sorts out languages into Germanic, Romance, Slavic, etc. Last but not least, the mathematician is always classifying--he sorts out numbers into even and odd; he categorizes geometric figures (by the number of sides) into triangles, quadrangles, pentagons, etc.; he distinguishes between rational and irrational numbers, and so on.

It is true that our penchant for classification is often misused and abused. Educators, for example, sometimes classify children into "bright" and "slow" when it is by no means clear that this kind of sorting out does not oversimplify a very complex situation and does not serve to bring useful insights. We are all aware of the mischief that simpleminded or even vicious stereotyping of people, ideas, or things can work. However, not only the legitimate uses, but even the abuses of identification schemes point to the importance of teaching children to become proficient at handling the ideas behind classifications. Indeed, if the elementary theory of sets has any place at all in the elementary mathematics curriculum, it is precisely because that theory helps us to understand the basic ideas of classification. It is to this purpose that the language of strings is utilized in the CSMP curriculum.

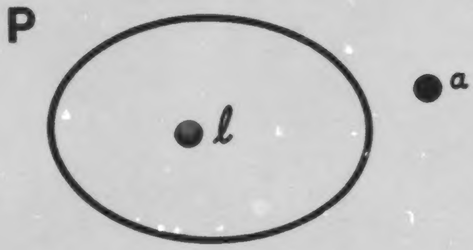
Classifying means just what it says--putting things into classes, or as the mathematician says: sets. In its simplest context a given thing either belongs to a given set (i.e., has a given property) or it does not. For example, since Abraham Lincoln was a President of the United States, he belongs to the set of American Presidents, while Spiro Agnew does not belong to that set. The mathematician abbreviates this by saying: Let P be the set of Presidents of the United Sates; let ℓ be Abraham Lincoln, let a be Spiro Agnew. Then,

$$\ell \in P \qquad \text{and} \qquad a \notin P$$

where the symbol "$\in$" is read "belongs to" or "is a member of". Similarly, if E is the set of even numbers and O is the set of odd numbers, we have

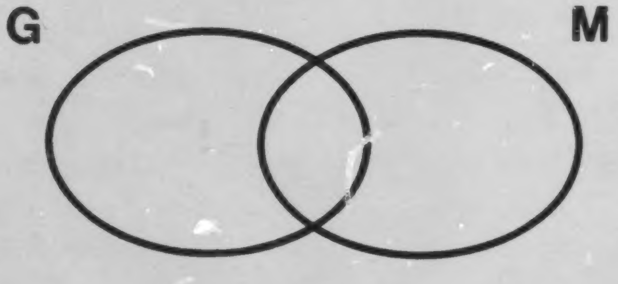
$$5 \notin E, \qquad 5 \in O, \qquad 6 \in E, \qquad 6 \notin O.$$

The basic idea is incredibly simple: Given a set S and any thing x, either x belongs to S or x does not belong to S; either x is in S or x is not in S; i.e., either $x \in S$ or $x \notin S$. It is the most basic act of sorting--"in" or "out". We can represent this elementary situation in a neat way by using the so-called "string pictures", which illustrate the idea of "in" or "out" in a very dramatic way. We represent the objects (the things) to be sorted by dots, and we represent the sets into which they are to be sorted by drawing "strings". Consider the Lincoln-Agnew situation referred to above. The string labeled "P" represents the set of Presidents. The dots represent various people. Lincoln is in P ($\ell \in P$) and Agnew is out; i.e., not in P ($a \notin P$).

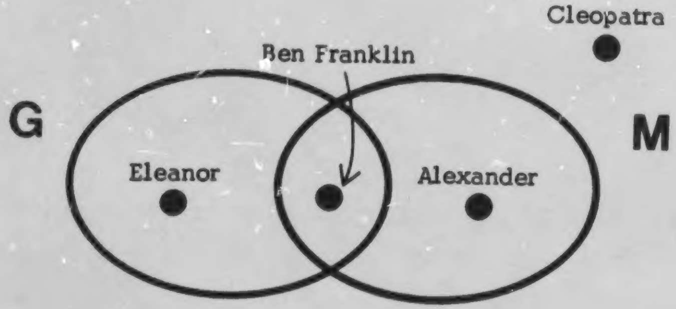


Note: A dot by itself can stand for any thing, e.g., Napoleon, the Eiffel Tower, the number 5, the city of Chicago, and so forth. But, once the dot goes inside the P-string it no longer can stand for just anything. It cannot represent the Eiffel Tower or Louis Armstrong; it must be for a President of the United States. Of course that still leaves (as of 1977) 38 choices (from George Washington to Jimmy Carter). Similarly, if a dot is outside the string it could be for the Eiffel Tower, or the Soviet Union, or the planet Mars, or even the number 5, but it cannot be for Thomas Jefferson.

The situation becomes interesting when we are sorting with respect to more than one property, i.e., when we are putting things into several "cubbyholes" (sets); because then we can study the relations between the sets and that is the beginning of the (very) elementary theory of sets. If G is the set of people who wear glasses and M is the set of males, we can draw two (colored) strings.



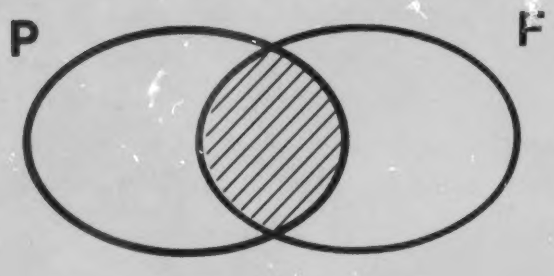
Now the classification scheme has to be concerned with two in/out decisions simultaneously. Consider Cleopatra--where does she go? She's not male and she did not wear glasses (glasses were not invented then!) so she goes outside both strings.



On the other extreme, Ben Franklin was a male who wore glasses, so he has to go inside both the M-string and the G-string. Alexander the Great, by the same argument as the one for Cleopatra, was a man who did not wear glasses so he goes inside the M-string and outside the G-string. Eleanor Roosevelt did wear glasses and was not a male, so she goes inside the G-string and outside the M-string. Note that every object x in the world has its place in the string picture above; there are four possibilities:

1. $x \in G$ and $x \in M$
2. $x \notin G$ and $x \in M$
3. $x \in G$ and $x \notin M$
4. $x \notin G$ and $x \notin M$

Of course, there are situations where it is impossible for anything to go into one of the "cubbyholes". For example, if P is the set of all U. S. Presidents and F is the set of all females, then we have (as of 1977) no x such that $x \in F$ and $x \in P$. We can indicate this state of affairs by hatching the appropriate region--to suggest that no dots should be placed in the region in question.



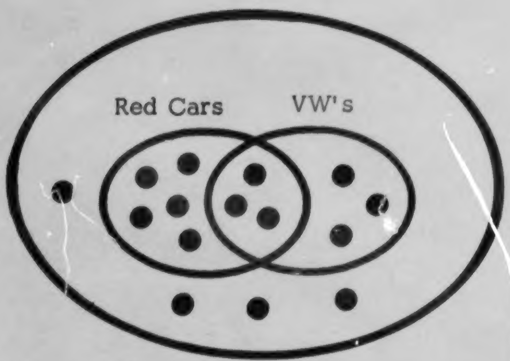
Thus the language of strings and dots gives children a precise (and nonverbal) way of recording and communicating their thoughts about classifying things. The abilities to classify, to reason about classification, and to extract information from a classification scheme are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. Ordinarily, people are not expected to deal with such problems until their command of their natural language is fully developed and until they face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires special modes of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification, thus freeing young minds (in some cases as young as four years of age!) to think logically and creatively about classes of things and to report their thinking long before they have the verbal skills that were thought to be required.

In this strand we set up situations and ask carefully phrased questions that continue to develop skills in classification, always remembering that the skills grow out of such experiences but that an ability to draw strings and arrows is not an objective in itself; the mode of thinking involved is the objective. Thus it is extremely important for us to set up our situations carefully. The objects we ask students to classify must have well defined attributes; otherwise, the students' attention will be distracted by the problem of deciding whether a certain object does or does not have a certain attribute. For example, consider this situation. A teacher suggests that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. She then invites a child, say John, to place a dot for himself on the diagram. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John does not think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such distraction and confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e., his classification of himself) and not on his decision as to what attributes he has.

It is quite clear, then, why we make such extensive use of A-Blocks (attribute blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, etc., and they know whether or not a given number of a set of numbers is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to be able to count the objects that have a certain attribute. For example, in a parking lot there are eight red cars and six Volkswagons. If there are four cars in the lot that are neither red nor Volkswagons, can we be sure there are 18 cars in the lot? Suppose we counted the cars in the parking lot and found there are exactly 15. Is this possible? A string picture immediately settles this apparent paradox.

Cars in Parking Lot



There are three red Volkswagons.

Relations: The Language of Arrows

In almost everything we do or think about we are intensely interested in the relations between things. This is true in our everyday lives, in school, in our careers, and in all scientific pursuits--we are always trying to establish, explore, and understand relations. Consider the following simple example. A girl comes running up to her mother and shouts: "Mommy, Mommy; Fido bit the mailman!" The girl is reporting to her mother that a certain relation has suddenly been established between the family dog and a certain mailman. Fido bit the mailman. The existence of the "biting relation" between Fido and the mailman has many important practical consequences. From the mailman's point of view, it is presumably a painful one and may lead him to consult a physician. From the mother's point of view, it is at the very least embarrassing, and might even lead to a potential law suit; from Fido's point of view, it will likely lead to punishment of some sort. For all these reasons and many others, the girl feels it urgent for the mother to know of the existence of the "biting relation" between the dog and the mailman.

To show our keen interest in relations and the great variety of relations that can exist between things, consider the following short list of everyday facts or statements. Each sentence asserts a relation between one object (person or thing) x and another y.

1. Johnny is taller than Mary. (establishes a relation between Johnny and Mary in terms of their height)
2. Johnny is Mary's brother. (establishes a blood relation between the two)
3. Johnny sent Mary a valentine.
4. Johnny borrowed $1 from Mary.
5. Johnny hates spinach. (a very common relation between a child and a vegetable)
6. Tootsie Rolls are 2¢ cheaper at Store A than at Store B. (establishes a relation between Store A and Store B in terms of their respective prices for Tootsie Rolls)
7. Giving is more blessed than receiving. (This example indicates that there are relations among relations. A certain relation (giving) is more blessed than another relation (receiving).)

Consider also the following sentences taken from various school subjects. Each sentence talks about a certain relation.

1. Mark Twain is the author of Tom Sawyer.
2. Shakespeare lived before Hemingway.
3. Huckleberry Finn is a sequel to Tom Sawyer.

Science:

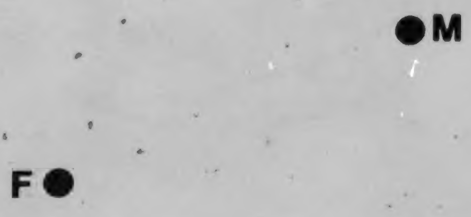
1. Lead is heavier than aluminum.
2. Aardvarks eat ants.
3. Phobos is a moon of Mars.
4. Malaria responds to quinine. (a relation between a disease and a certain drug)

History:

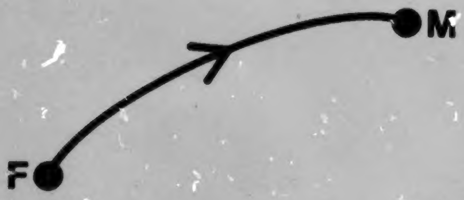
1. Brutus killed Julius Caesar.
2. Louis XIV was King of France. (a relation between a person, Louis XIV, and a country, France)
3. Hitler hated Jews. (a relation between a person, Hitler, and a class of people, the Jews)
4. President Kennedy was assassinated in Dallas. (a relation between a person and a city)

Let's hope that the examples above have made our main point: we are interested in the relations between things. Mathematics is no different from everyday life, English literature, science, or history. To study mathematics means largely to study the relations between mathematical objects.

As always, if we are to study anything seriously, we will need a *language*, a way of symbolizing the things we want to study. A very apt language indeed for studying and talking about relations is the *language of arrows*. This is the language that is exploited systematically throughout this strand for thinking and writing about relations. Let's quickly remind ourselves how the language of arrows works. We have already agreed earlier to represent objects by dots. In the story of Fido biting the mailman, we would represent both Fido (F) and the mailman (M) by dots.

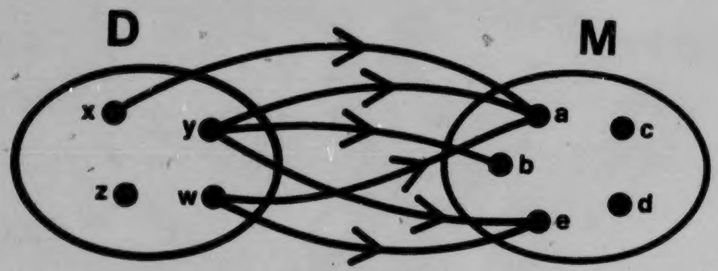


What could now be more natural to represent the fact that Fido bit the mailman than an arrow from Fido to the mailman.



In fact, with a little imagination the arrow suggests to our mind's eye quite vividly Fido's rushing towards the mailman and biting him. Note that, of course, we have drawn the arrow from Fido to the mailman. The reverse arrow (from the mailman to Fido) would mean the mailman bit Fido (in the spirit of the classical "man bites dog" saying!).

In general, if we are trying to picture any relation R between objects (biting, being the brother of, sending a valentine to, borrowing $1 from, and so on) we draw an arrow from object x to object y whenever x relates in the specified way to y. Suppose, for example, D is a set of dogs and M a set of mailmen, and the red arrows represent the relation: bites.



Note how very vividly our arrow diagram presents the state of affairs. The answers to the following kinds of questions can be read off from the diagram almost instantly.

Which dog seems to be the most vicious?

Which dog seems to be the gentlest?

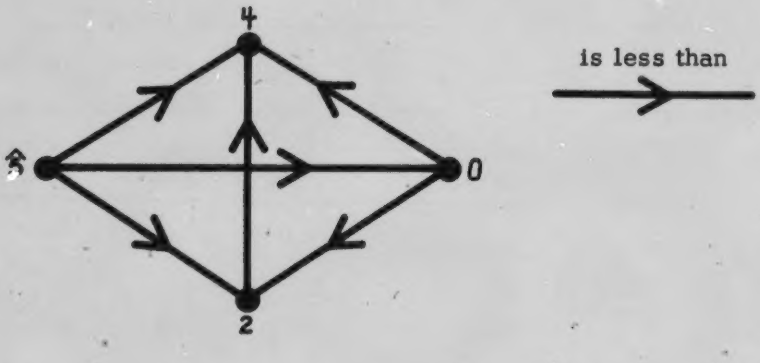
Which mailmen seem to get along best with the dogs?

Of course, frequently we are interested in comparing and contrasting two or more relations simultaneously. In that case we simply draw two superimposed arrow diagrams and code the arrows by color to show which arrows go with which relation.

In summary: Arrow diagrams are a very handy graphical representation of a relation, somewhat the same way that a blueprint is a very handy graphical representation of a house. By means of arrow diagrams we can represent important facts about a given relation or relations between objects in a very simple, suggestive, pictorial way--usually much more conveniently than the same information could be presented in words. The convenience of arrow diagrams has of course important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read"--and also draw--an arrow diagram of a relation long before he or she could read or present the same information about that relation in words or other symbols. The situation here is the same as in so many other cases. The problem of presenting certain ideas to children lies not in their intellectual ability to grasp the ideas; rather, the limitations are often just mechanical, motor problems of writing or reading about the ideas in conventional systems. The arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw.

In the discussion above, we introduced and explained the arrow diagrams in terms of a relation that comes from everyday life: dog bites mailman. Obviously, in the mathematics class we are also interested in relations between mathematical objects like numbers, lines, triangles, outcomes of experiments, and so on. However, we showed how arrow diagrams can be used to picture nonmathematical relations just as well as mathematical ones--for an important reason: to show that the mathematics class is not an isolated world in which we play exotic games totally different from what we do in the rest of our lives.

The important point here is simply this: thinking about and exploring the world of numbers is an activity very similar to thinking about and exploring any other world, e.g., the world of dogs and mailmen. Moreover, the tools we use to explore and understand the everyday world (e.g., arrow diagrams) are useful in exploring and understanding the world of mathematics. Conversely, the tools we develop and use to help us think about mathematical things very often help us in thinking about other kinds of things also. To see the analogies, consider the following simple arrow picture where a red arrow from x to y means x is less than y.



To make our point more graphic, we imagine the numbers (which are, after all, things) talking to each other via the red arrows and saying: I am less than you.*

Now we can ask such questions as:

1. Who does the most talking (i.e., which is the smallest number)?
2. Who does the most listening (i.e., which is the largest number)?

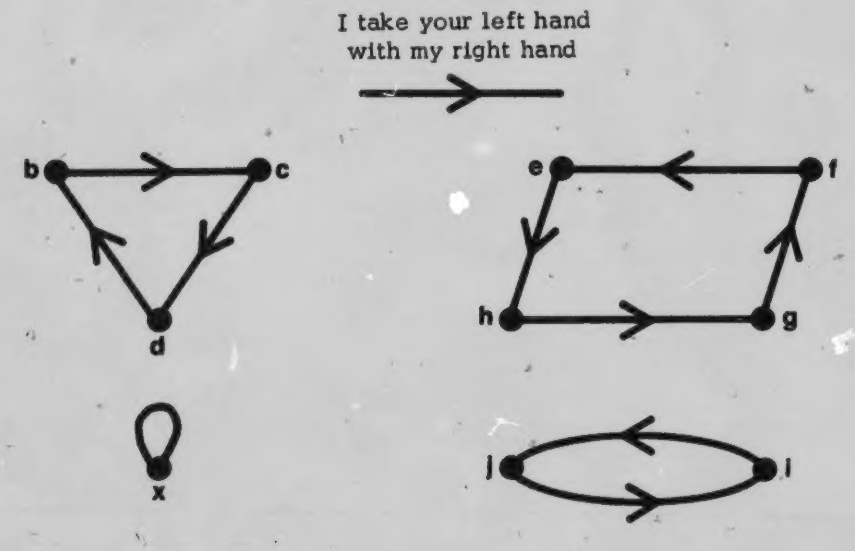
We leave it for the reader to find questions analogous to (1) and (2) above in the dogs and mailmen situation discussed previously.

---

* Aesop, to make his points, used talking wolves and foxes and dogs. In the same spirit, we like to have the numbers talk. After all, the number 2 probably knows more about its relations to other numbers than anyone else, so why not have it tell us what it knows?

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations between things. Again it must be remembered that our goals concern the thinking process and not the mechanism. Thus we must resist making the ability to draw prescribed arrows an objective in itself, just as we must resist viewing an arrow diagram as just another format for drill problems in arithmetic.

There is another educational bonus that occurs when arrow diagrams are used imaginatively. A story can be captured and described by an arrow diagram, and conversely, an arrow diagram can suggest many possible stories and games, allowing children to expand their creativity in many ways. In this sense, an arrow diagram resembles a script for a play or a choreographer's dance diagram. For example, suppose each child in the class is told to do the following: With your right hand take the left hand of one child. What confusion! When the instructions are eventually carried out they can easily make a record of this event (or it might even turn out to be a dance) with an arrow diagram that might look like this:

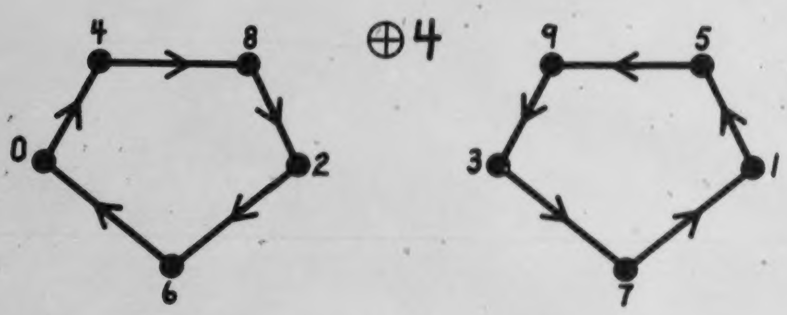


(Notice that student x took the easy way; he took his own left hand with his right hand.)

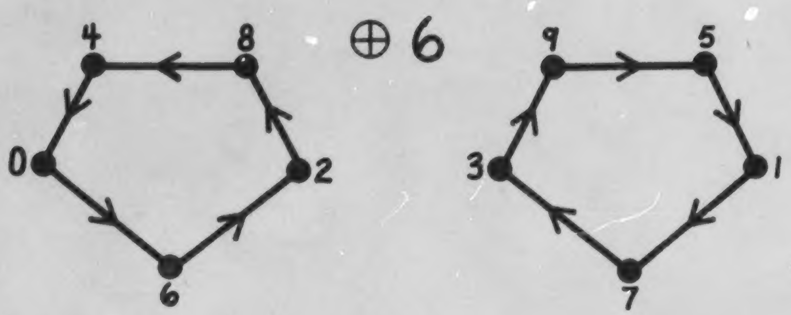
Conversely, given such an arrow diagram (which describes a relation called a permutation, because every dot has exactly one arrow starting from it and exactly one arrow ending at it), the children can use it as a "dance plan" and follow its directions in the next "dance". Of course, the same diagram might suggest a wide variety of stories that children can tell, such as:

> "On Christmas Day, everyone in our big family gave exactly one gift to someone in the family and received exactly one gift from someone (not necessarily the same person to whom his gift was given) in the family. Al (denoted "x" in the diagram) knew what he wanted, so he gave his gift to himself. Beth (b) gave a doll to Clara (c), ..."

It is not surprising that such arrow diagrams, and the games suggested by them, yield important mathematical concepts. Consider, for example, the following story situation in which the numbers 0, 1, 2, ..., 9 become separated from the rest of the whole numbers and are looking for a game to play. They decide they cannot play their usual addition games, for that would require all the whole numbers to be present. So the ringleader, Zero, suggests a new "addition" game, ⊕, that they can play, in which 1 ⊕ 7 = 8, 3 ⊕ 6 = 9, 2 ⊕ 8 = 0, 7 ⊕ 6 = 3, etc. Now they can all play the game because they use only the last digit of the ordinary sum for this new kind of sum. "Let's try the ⊕4 snake dance!" shout the numbers. And they do.



"Also, the ⊕ 6 snake dance!"



"The same dance, but in the opposite direction," they notice.

The arithmetic of the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} determined by this new operation, ⊕, illustrates some very basic concepts in mathematics, but at this level it is just one more interesting experience that generates good mathematical thinking.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze given situations.

By this time your students should be very familiar and quite comfortable with both the language of strings and the language of arrows. The content of this strand for the present semester builds in a variety of ways on this fluency with the languages. Each of the two chapters included in this strand present situations quite distinct from the other.

Chapter 1 continues the work begun in the chapter on Composition Games (Chapter 3 in The Languages of Strings and Arrows strand) from CSMP Mathematics for the Intermediate Grades, Part I. Here the internal structure of the language is exploited to pose situations involving the composition of relations. The situations in this chapter very often lend support to and continue the work begun in The World of Numbers strand on the composition of numerical relations.

The lessons in Chapter 2 extend the logical thinking lessons from Part I (Chapter 2 in The Languages of Strings and Arrows strand) and begin a systematic analysis of a mathematically rich game (The String Game) which has become very familiar to the students.

1

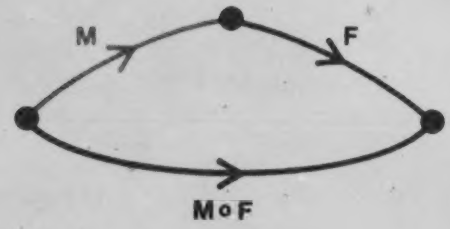
RELATIONS, FUNCTIONS, AND COMPOSITION

In Part I of the CSMP Intermediate Grades Program, the concept of composition of relations was explored through a series of games in which the relations were not specified but were identified only by colors. You may find it helpful to reread the introduction to Chapter 3, Composition Games, on page 77 of the Languages of Strings and Arrows strand in the Part I teacher's guide.

In Section 1.1 we review the composition of two unspecified relations denoted by letters, in this case " R " and " **S** ". The composite relation, R followed by S, is drawn in red and denoted "**R o S**". It is natural for the students to read the symbol, "R • S", as "R circle S", and this should be encouraged. Of course, any reasonable reading should be accepted, such as "R then S", "R followed by S", or "R, S".

NOTE: Some mathematicians write "S • R" for "R followed by S", and others write "R • S", as we do here. A choice has to be made; we selected what seems to be most natural for our purposes.

After reviewing the idea of composition of relations, the first lesson continues with some specific interpretations of the relations; for instance, some "family" relations such as "you are my father" and "you are my mother", and some numerical relations, such as +2, ×3, etc. In this manner, composition of relations begins to take on concrete meaning not only in an unspecified set of arrows but in familiar settings of family relations, arithmetic, etc. For example, if M is the relation "you are my mother" and F is the relation "you are my father", then M • F is the relation "you are my maternal grandfather".

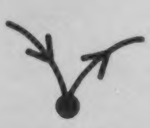


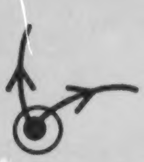
The second lesson introduces a special kind of relation, namely, a function. Unfortunately, the concept of function is usually encountered in school experience in terms of formal symbolism that obscures its simple meaning. A function is any relation for which at most one arrow of the relation starts at each dot.

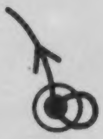
This extremely important, but simple, idea is presented in a game in which students are invited to draw a blue relation with many arrows but following this rule:

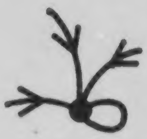
at most one blue arrow may start at each dot.

Dots at which two or more arrows start are called "guilty". So students begin to think of functions as relations with no "guilty" dots.


not guilty

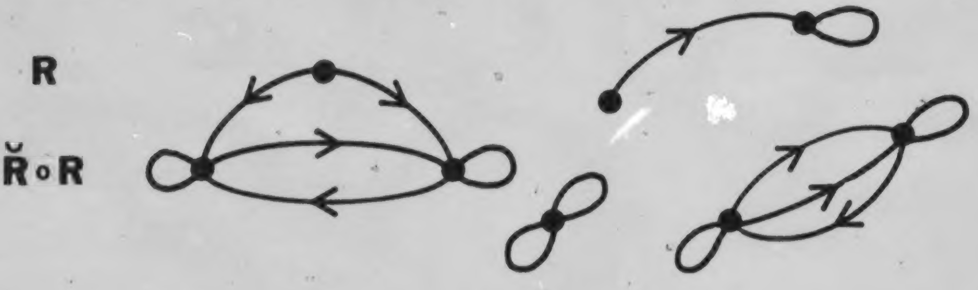

guilty


guilty

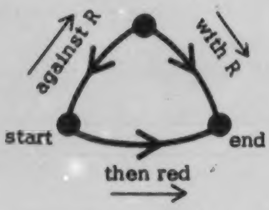

not guilty

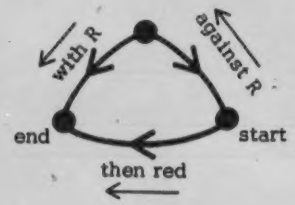
not guilty

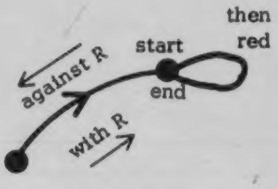
In the third lesson, for the first time the idea of composition of relations is combined with the converse of a relation. The words used here, "composition" and "converse", are not used with the students. These are sophisticated words referring to simple ideas. Given a relation R, the converse $\check{R}$ of R is simply the relation consisting of all the return arrows of R. We will use the language "against R" for "the converse of R", because it conveys the idea of return arrows. For example, if the blue relation in the arrow picture below is R, then the red arrows are for the relation $\breve{R} \bullet R$.



"$\breve{R} \circ R$" is read "against R followed by with R" or "against R circle R". Notice how graphic the words "against" and "with" can be when tracing the arrows.



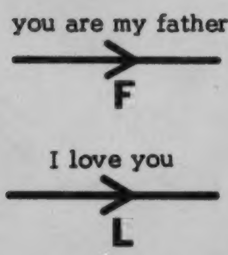


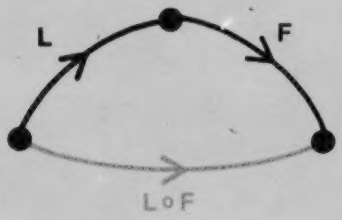


With practice the students begin to find $\breve{R} \bullet R$ arrows (against-with) as well as $R \bullet \breve{R}$ arrows (with-against). Some begin to discover that there is an $\breve{R} \bullet R$ loop at the end of each R arrow, and there is an $R \bullet \breve{R}$ loop at the start of each R arrow.

Caution: These experiences with "against-with" and "with-against" arrows are not intended to lead to mastery of any skills or techniques. They should be enjoyable experiences and exploratory experiences. Drill on these ideas would defeat the purpose, which is to give a first taste of a concept that later will be useful in understanding more concrete portions of mathematics.

The remaining lessons of this chapter are designed to involve the students in interpreting composites of familiar relations such as "I love you" and "you are my father". Students consider the arrow pictures of composite relations as messages to be decoded. For example, here the yellow arrow denotes L • F,





which can be decoded as the message "you are the father of someone I love". Similar messages can be written by the students for decoding by other students, and also can be analyzed to determine, for example, how many messages are possible with two different relations on three dots.

The familiarity of the situations that are expressed by such messages make the experience not only effective in teaching the idea of composition of relations, but also affective in terms of students' emotional involvement.

CAPSULE LESSON SUMMARIES

Section 1.1:	Given an arrow picture involving two unspecified relations and given a rule for a composition of the relations, draw all the arrows for the composite. (Worksheets L01, L02, L03, L04)
Section 1.2:	Consider special relations called functions; that is, relations for which at most one arrow starts at each dot. Determine whether given relations are functions. (Worksheets L05, L06)
Section 1.3:	Consider composites of a relation R with itself ($R \cdot R$), of R with the converse of R ($R \cdot \check{R}$), and of the converse of R with R ($\check{R} \cdot R$), using the language "with R" (for R) and "against R" (for $\check{R}$). (Worksheets L07, L08, L09, L10)
Section 1.4:	Identify "messages" involving the relations "you are my father" and "I love you", and interpret various composites of these relations and their converses.
Section 1.5:	With two relations specified, continue to interpret certain composites. Given a description of a composite in English, determine the arrow diagram for the composite. (Worksheets L11, L12, L13, L14, L15, L16)

1.1 Composition of Relations

Exercise 1

On page 10 you will find an arrow picture that is the basis for this lesson. According to your preference, you can either carefully copy this picture on the board before the lesson begins, or you can display one copy of the IG-II, Language Poster #1 by taping it to the board. A second copy of this poster is provided in case a replacement is necessary. If you use the poster, draw the red arrows with a red felt marker or a bright red crayon.

Trace several yellow arrows in the picture and point to " R " as you say:

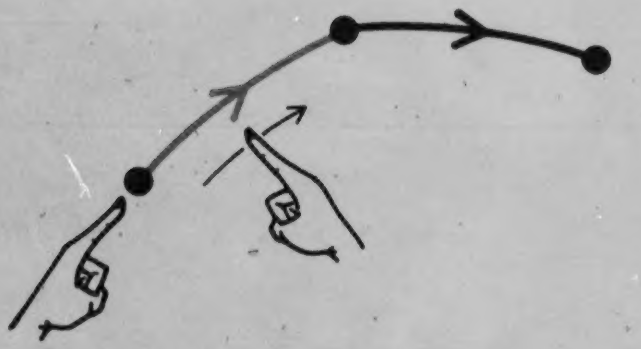
T: The yellow arrows are a relation we call "R".

Then trace several blue arrows and point to " S " as you say:

T: The blue arrows are a relation we call "S". Now we are going to use red arrows for the relation "yellow followed by blue". Here is the rule for drawing red arrows . . .

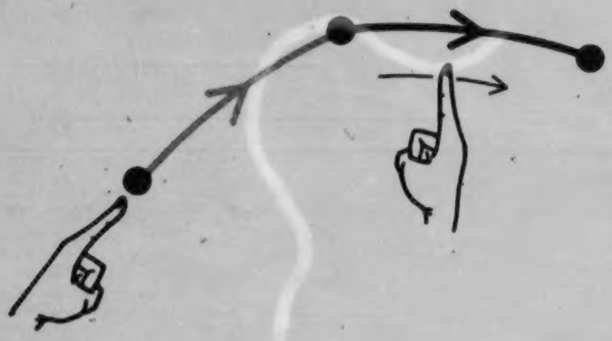
Use the upper left corner of the arrow picture for your explanation. As you explain the rule stated in the left column below, make the motions described in the right column.

T:	Each time there is a dot from which a yellow arrow starts . . .	Place your left forefinger on a dot at which a yellow arrow starts. Hold it there. With your right forefinger trace the yellow arrow in the direction of the arrowhead.



T: . . . and the yellow arrow ends at a dot from which a blue arrow starts . . .

Stop the motion of your right forefinger at the middle dot, tap the dot, and then trace the blue arrow.



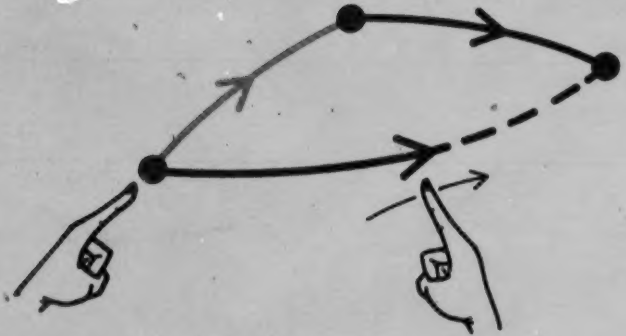
T: . . . then we must draw a red arrow from the dot where the yellow arrow starts . . .

Tap this dot several times with your left forefinger.

. . . to the dot where the blue arrow ends.

Tap this dot several times with your right forefinger.

First trace and then draw the red arrow.



Repeat the series of motions above, saying "yellow . . . followed by blue . . . is red".

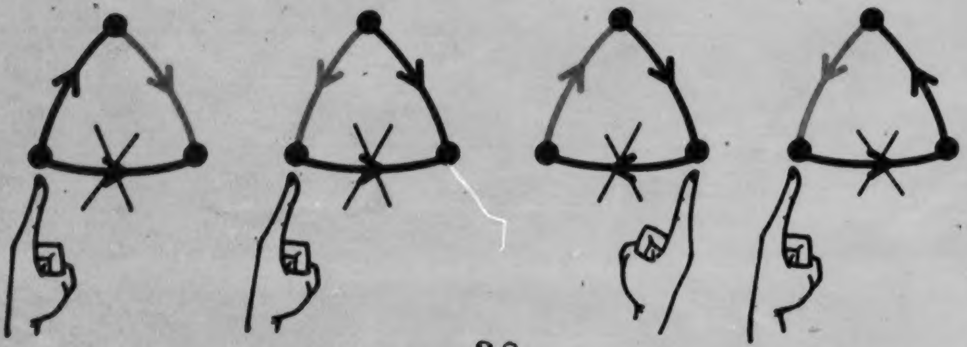
T: There are many more red arrows we can draw. Who can see where there should be another red arrow?

Let the students take turns coming to the board and showing where a red arrow should be drawn. Ask each student to first trace the arrow and then, if correct, to draw it in red.

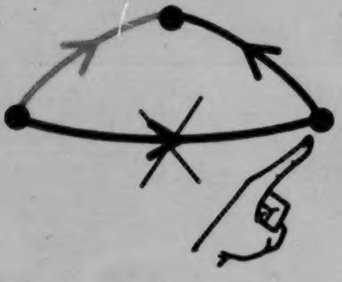
Note on common mistakes and difficulties: Students who have difficulties - common mistakes are illustrated below - may be helped by asking one of these questions.

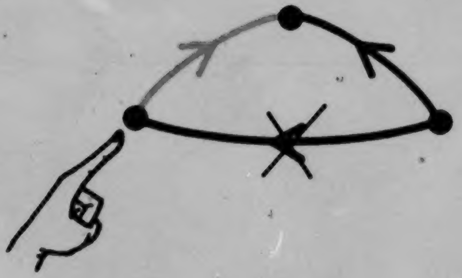
T: Where should a red arrow start? (At a dot where a yellow arrow starts)

Does a yellow arrow start here? (No)

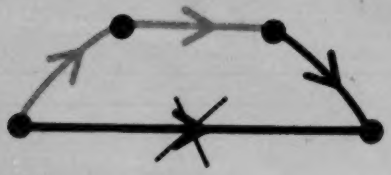


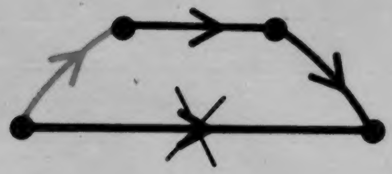
T: Where should a red arrow end? (At a dot where a blue arrow ends) Does a blue arrow end here? (No)



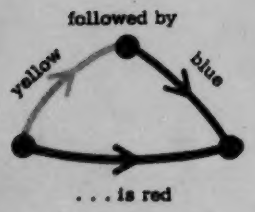


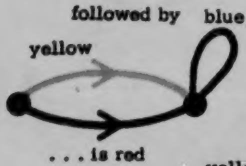
T: How many yellow arrows should be followed by how many blue arrows? (One yellow arrow followed by one blue arrow, with a connecting dot where the yellow arrow ends and the blue arrow starts) Is there one yellow arrow followed by one blue arrow? (No)

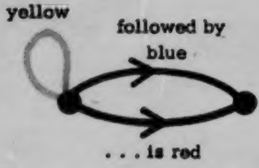


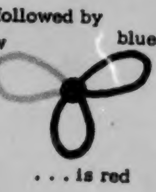


Encourage the students to say "yellow followed by blue is red" as they trace the arrows. The following diagrams show various ways red arrows will be found in the arrow picture.


followed by
yellow
blue
. . . is red


followed by
blue
yellow
. . . is red

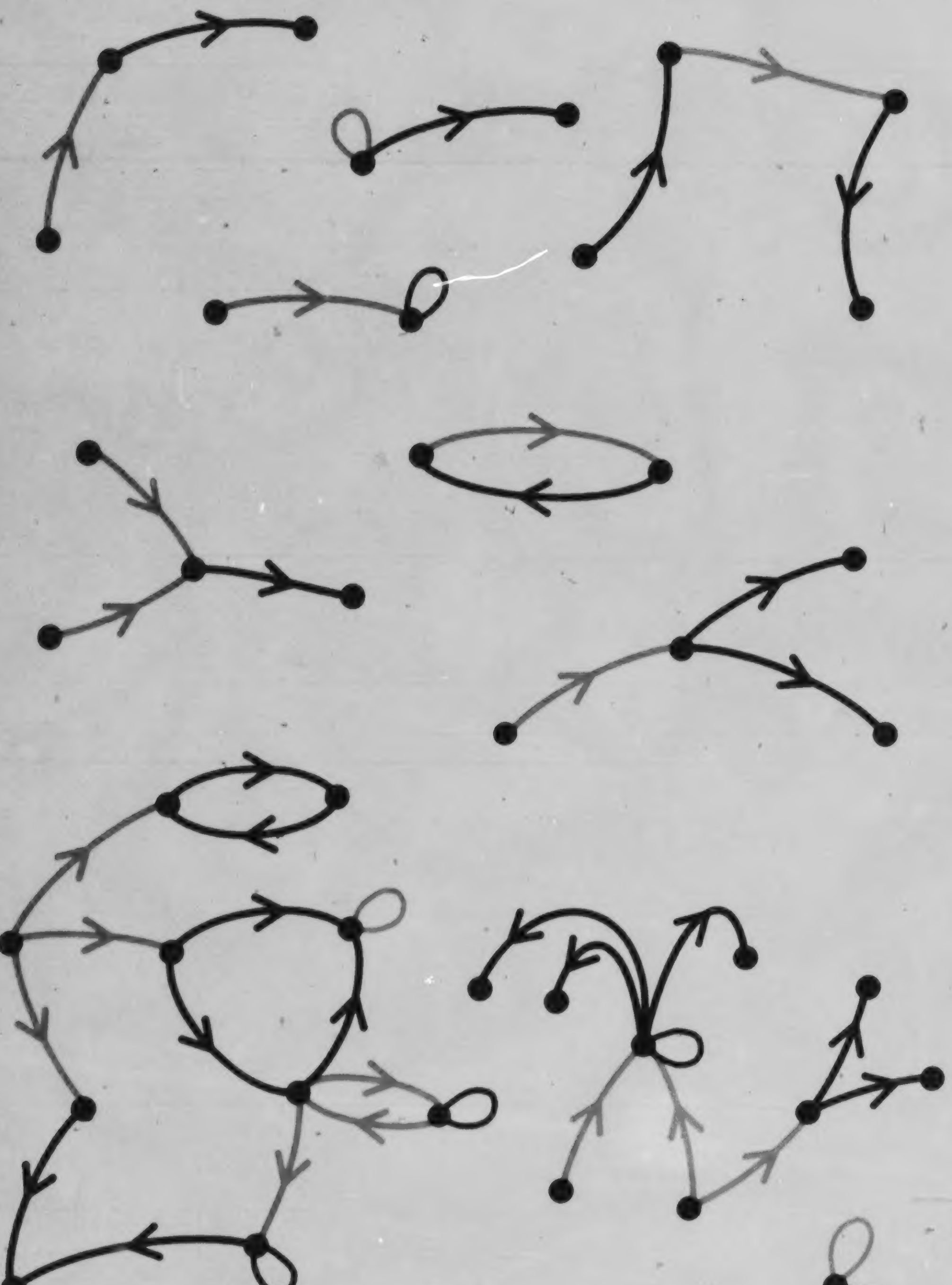

yellow
followed by
blue
. . . is red


followed by
yellow
blue
. . . is red

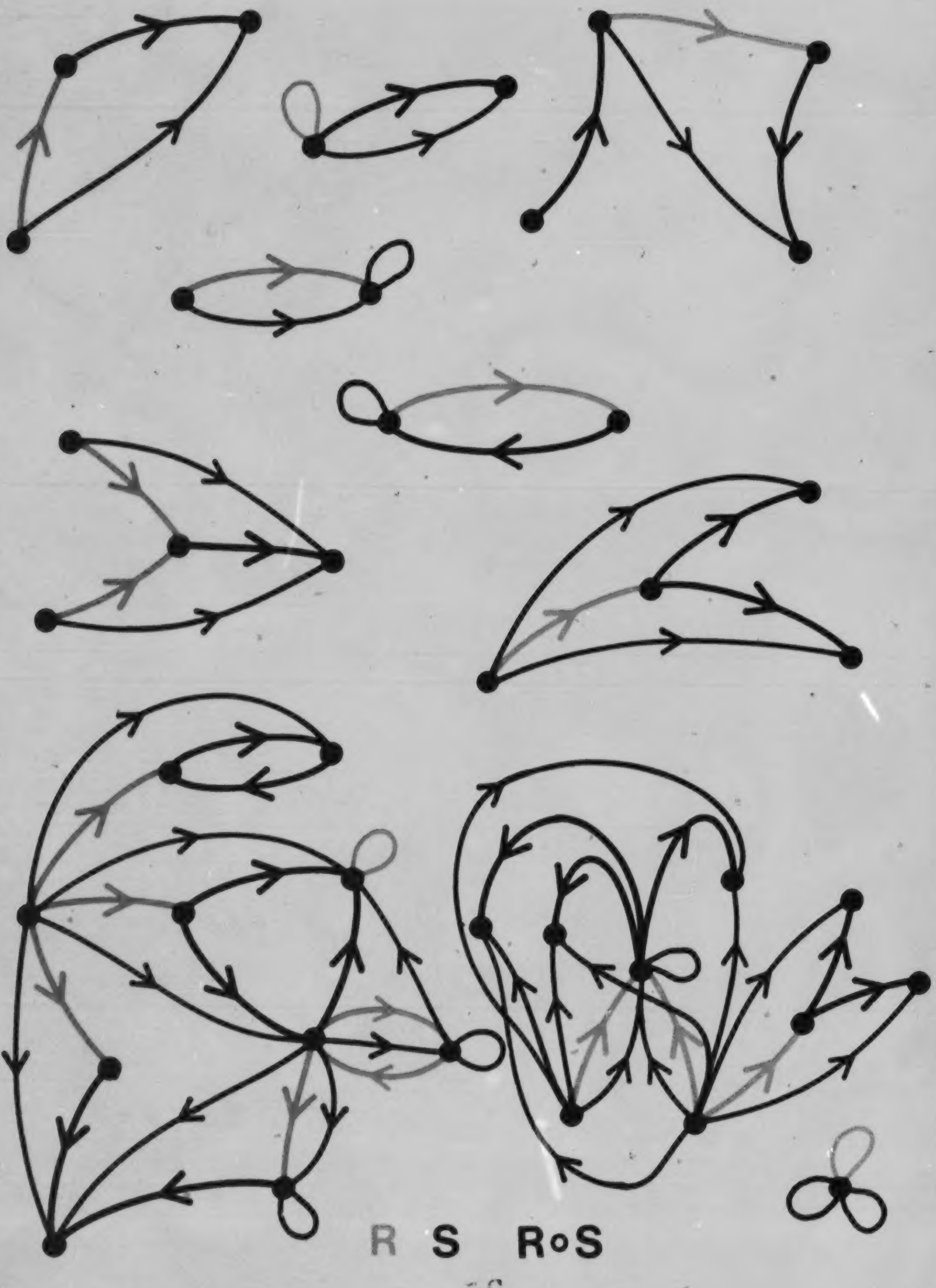
Every student should be able to participate in this activity, and it should move quite quickly. When most of the red arrows have been drawn and the picture is rather complicated, it may be difficult to see whether or not there are more red arrows to draw. However, with a little encouragement, your class should be able to find them all.

After all the red arrows are in the picture (see completed picture on page 11), remind the class that we called the yellow relation "R" and the blue relation "S". Explain that we will call the red relation "R • S" (read this "R followed by S") and write "R∘S" at the bottom of the picture. Your students may have other ways of reading this symbol; you should accept any reasonable reading such as "R then S", "R circle S", or "R, S".

1L

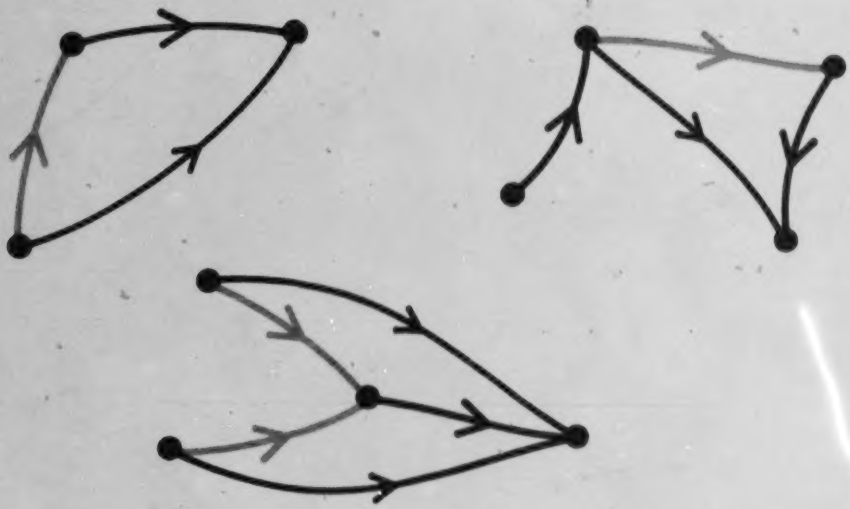


Answer key for the red relation R ∘ S



Exercise 2

Erase the relation names and completed arrow picture except for this part. If you used a poster, take it down. Copy this picture on the board.



T: Let us suppose the dots are for some of the people at a family reunion. What could the yellow relation be?

Allow the students to make many suggestions such as: "you are my brother", "you are my mother", "you are my uncle", etc.

T: Let's say that the yellow relation is "you are my mother"; and to remind us we will call it "M". (Write " M " on the board.) Which people are saying to other people "you are my mother"?

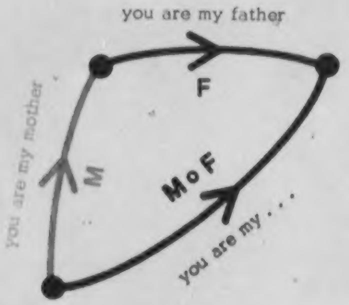
Have students point to the dots for these people in the picture.

T: Let's say that the blue relation is "you are my father"; we will call it "F". (Write " F " on the board.) Can someone tell us what the red relation could be for?

S: M circle F (or M followed by F).

T: How do we describe the M • F relation at a family reunion?

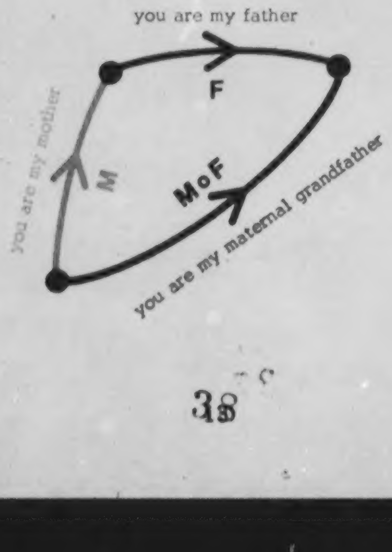
To help the students think about this, label the arrows in the upper left part of the picture.


you are my father
F
you are my mother
M
MoF
you are my . . .

S: You are my mother's father.

S: You are my grandfather.

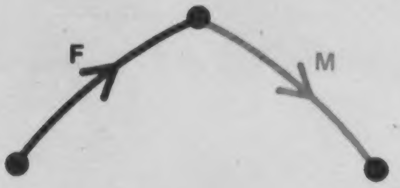
T: Yes, it could be "you are my grandfather", but remember that you have two grandfathers: one is your father's father and the other is your mother's father. The red relation is "you are my mother's father", or saying it another way, "you are the father of my mother", or to use a special English word, "you are my maternal grandfather".


you are my father
F
you are my mother
M
MoF
you are my maternal grandfather

T: If M • F is the relation "you are my maternal grandfather", what is the relation F • M?

Write "F • M" in white or any different color chalk on the board. Black is used in the illustrations here. Also, add these arrows to your picture.

F ∘ M



S: You are my father's mother.

S: You are my grandmother.

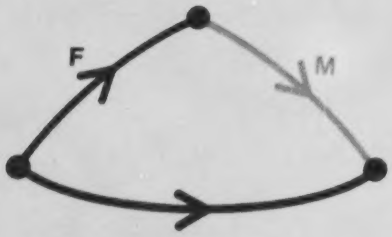
S: You are my paternal grandmother.

You might have to introduce the word "paternal" to the class.

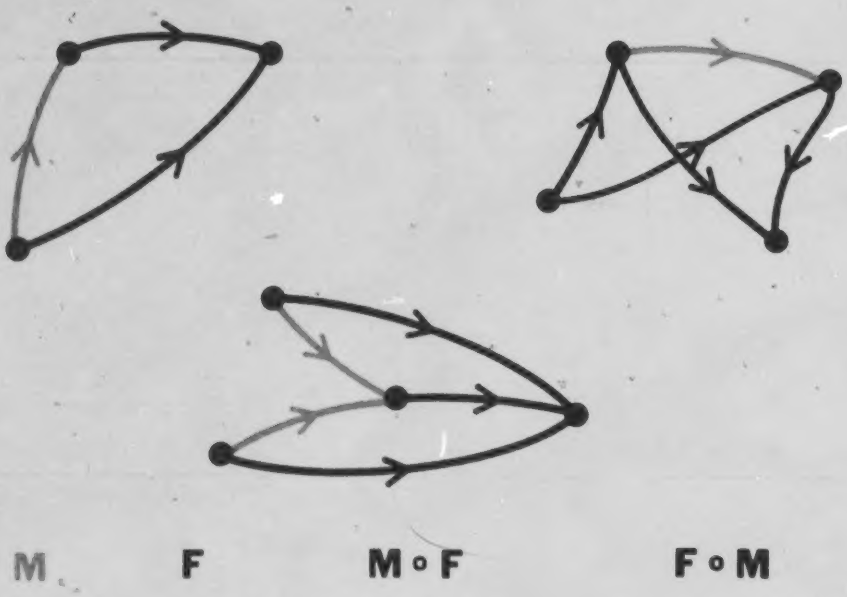
T: Can someone draw an F • M arrow in this picture (point to the part you just added)?

Help a student to trace F . . . followed by M . . . is F ∘ M, and then to draw the F • M arrow.

F ∘ M



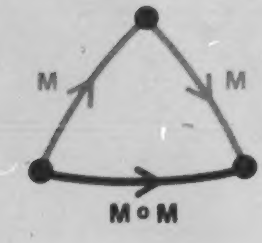
T: Do you see any F • M arrows we could draw in our family reunion picture?



The class should find that there is only one F • M arrow to be drawn.

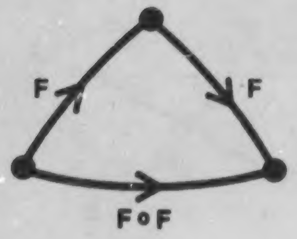
If this activity is still lively, ask for the meanings of some other composite relations.

NOTE: Each of the composite relations illustrated below are in red. You should try to use a different color for each when you draw them on the board.

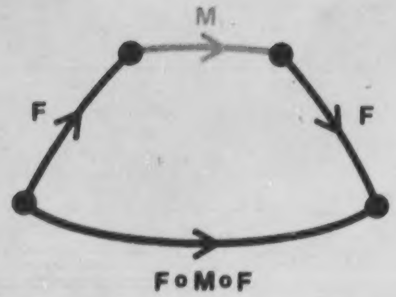


M • M (Answer: "you are my mother's mother" or "you are my maternal grandmother")

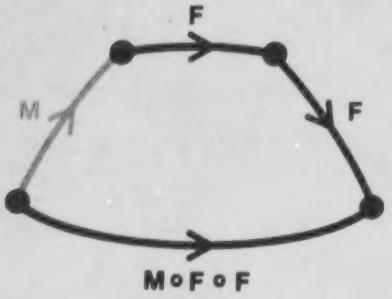
1L



F • F (Answer: "you are my father's father" or "you are my paternal grandfather")



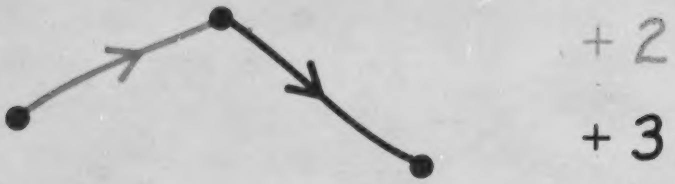
F • M • F (Answer: "you are my father's maternal grandfather")



M • F • F (Answer: "you are my mother's paternal grandfather")

Exercise 3

Erase the board and draw this picture.



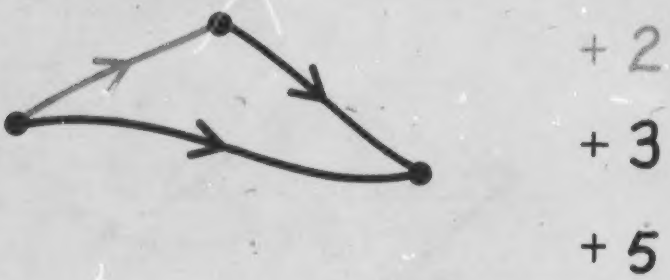
T: This time the dots are for numbers and the yellow arrows are for the relation + 2 and the blue arrows are for the relation + 3. What is the relation (+ 2) • (+ 3)?

S: + 2 followed by + 3.

S: + 5.

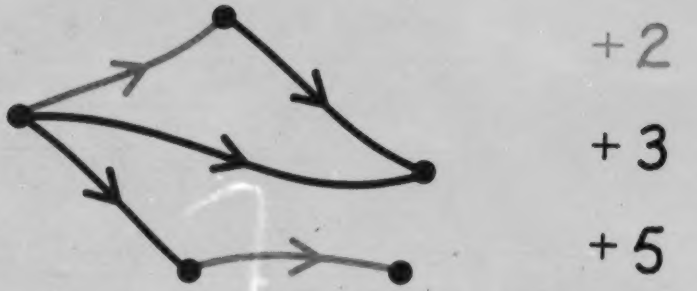
Let a student draw a red arrow in the picture for + 5, while you write:

$$(+2) \circ (+3) = +5$$



T: How about (+ 3) • (+ 2)?

As you ask this question, add arrows to the picture on the board, as shown below.



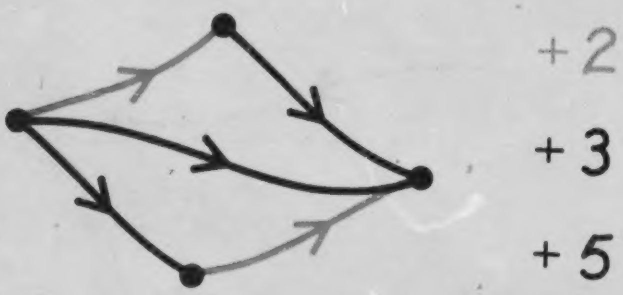
S: + 3 followed by + 2 is the same; + 5.

T: How should I fix the picture?

S: The bottom yellow arrow must end where the red arrow ends. + 3 followed by + 2 is the same as + 2 followed by + 3.

Do this on your picture and write:

$$(+2) \circ (+3) = +5 = (+3) \circ (+2)$$



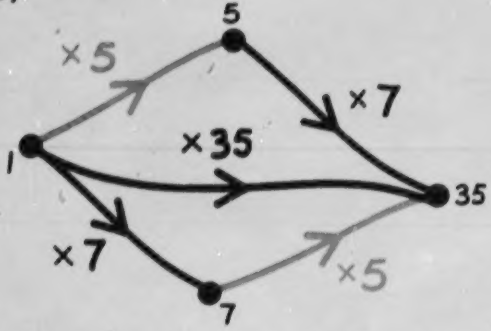
Repeat this exercise for

$(\times 2) \circ (\times 3) = ?$ (Answer: ×6) $(\times 3) \circ (\times 2) = ?$ (Answer: ×6)

and

$(\times 5) \circ (\times 7) = ?$ (Answer: ×35) $(\times 7) \circ (\times 5) = ?$ (Answer: ×35)

If any of these problems cause difficulty, you could label a dot with some number and have the students label the other dots before deciding on the composite relation. For example,

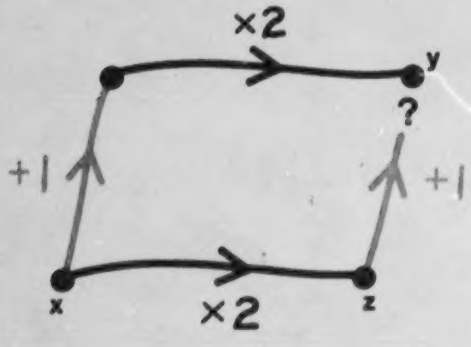


$$(\times 5) \circ (\times 7) = \times 35 = (\times 7) \circ (\times 5)$$

Exercise 4

Put this problem and arrow picture on the board.

$$(+1) \circ (\times 2) \overset{?}{=} (\times 2) \circ (+1)$$



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Is it true that $(+1) \circ (\times 2) = (\times 2) \circ (+1)$?

Let the class discuss the problem. Students might suggest starting with some number for the dot labeled "x" and checking. They will probably decide that the relation (+1) • (×2) is not the same as (×2) • (+1), but might not be able to explain why.

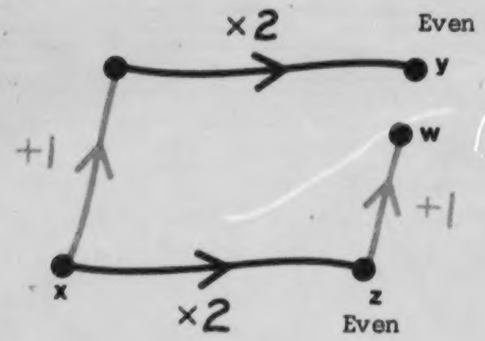
T: Is this number (point to y) even or odd? Why?

S: Even, because it is the double of a number.

T: Is this number (point to z) even or odd?

S: Even.

Add the dot labeled "w" to your picture.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

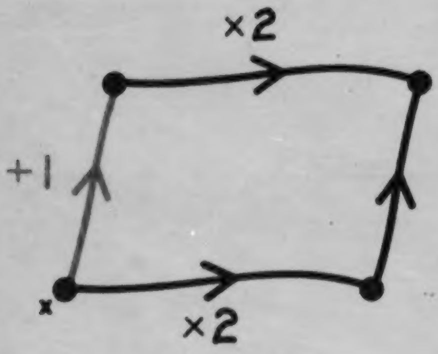
T: Then is this number (point to w) even or odd?

S: Odd; it is one more than an even number.

T: Can a number be both even and odd? (No)

So (+1) • (×2) cannot be the same as (×2) • (+1).

Draw this red arrow.

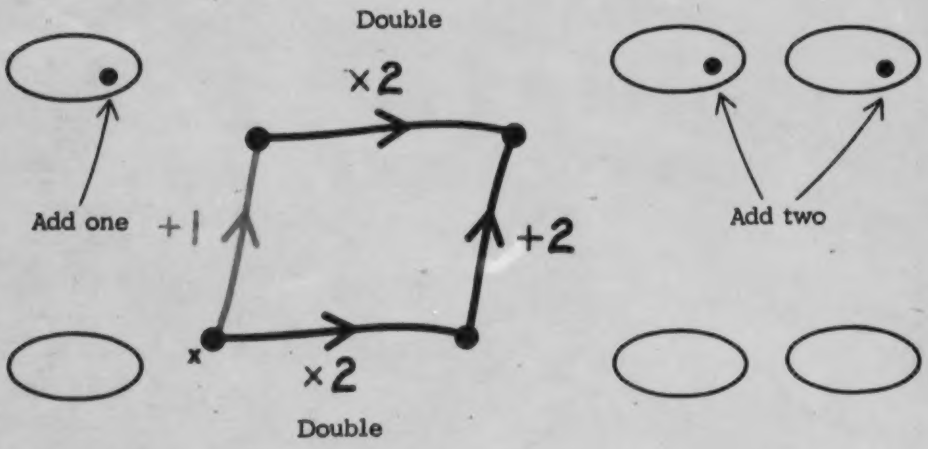


Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: What could this red arrow be for?

S: + 2.

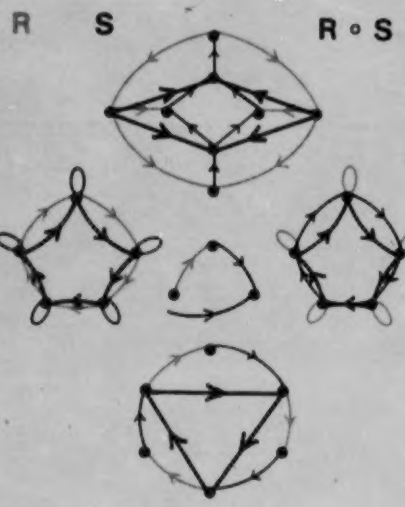
If no explanation is offered, you could try illustrating the situation by starting with some number of "invisible" dots at x. Here "invisible" is meant to suggest that we could start with any number at x.



Worksheets L01, L02, L03, and L04 are available for the rest of the lesson.

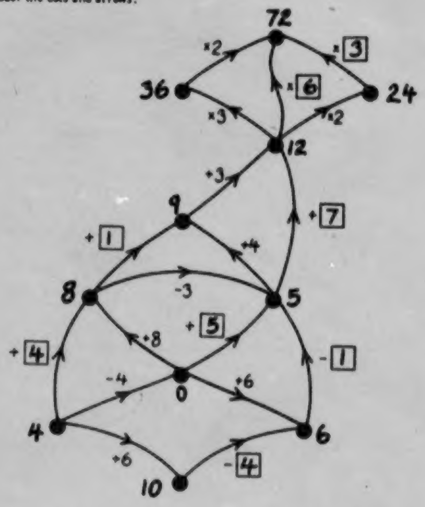
Name ____________________ L 01 *

Draw the missing red arrows.



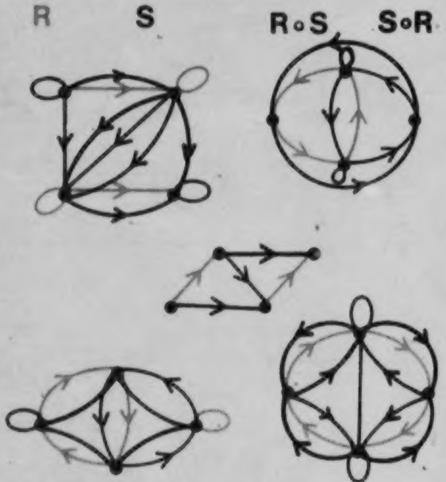
Name ____________________ L 02 **

Label the dots and arrows.


72
×2
×3
36
×6
24
×3
×2
12
+3
9
+7
+1
+4
8
-3
5
+5
+8
+4
-1
-4
0
+6
4
6
+6
10
-4

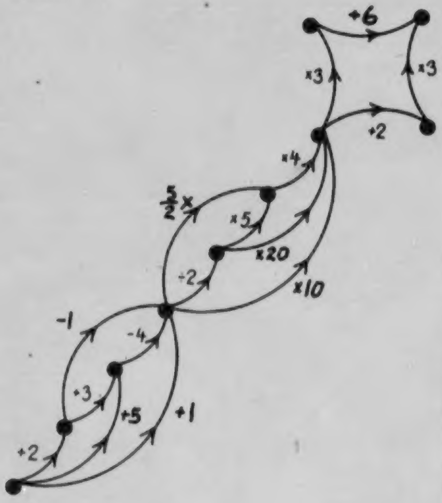
Name ____________________ L 03 ***

Draw the missing red and black arrows, loops, and cords.



Name ____________________ L 04 ****

Label the arrows.


+6
×3
×3
+2
×4
$\frac{5}{2}$×
×5
×20
÷2
×10
-1
-4
+3
+5
+1
+2

1.2 Functions

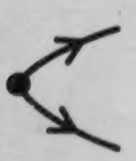
Exercise 1

Distribute Worksheet L05 and explain to the class that they may draw any blue relation they wish, but they must follow the rule:

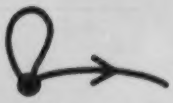
at most one blue arrow starts at each dot.

You may find it necessary to remind the class of the meaning of "at most one". Let the students discuss the phrase and use it in several different contexts. They should understand that "at most one" means "one or none" or "no more than one".

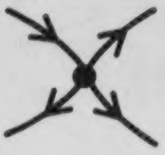
Encourage the students to draw a beautiful blue arrow picture, and with each new arrow they draw to check the "at most one" rule. As you walk around the classroom, check that the rule is being followed. If you see an error, such as



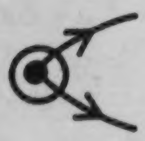
or

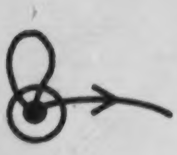


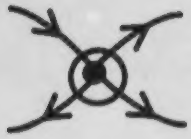
or



circle the dot with a red pencil







and say, "Here is a guilty dot."

A student might suggest that he or she need not draw any blue arrows to follow the rule. That would be correct; it is a function with no arrows. In this case, suggest that the student try to draw a more interesting picture.

When many students have drawn elaborate pictures and you have circled as many "guilty" dots as you can find, stop this activity.

T: If you violated the rule for the blue arrows and have drawn two or more blue arrows starting at one dot, that dot is "guilty" and I circled it in red. Did anyone draw a blue relation with no guilty dots? Then you have drawn a function.

Write this definition on the board.

FUNCTION

A function is a relation with at most one arrow starting at each dot.

T: How many dots are there on our worksheet? (25) What is the largest number of blue arrows you could draw and still have a function? (25) Why? How would you draw them?

S: From each of the 25 dots you can start one blue arrow, but no more.

T: What is the smallest number of arrows you could draw for a function?

S: 0. You could have no arrow starting at each dot.

T: Yes. That would be a function.

Exercise 2

Draw about 30 dots on the board.

T: I have drawn a function; a rather unexciting one with no arrows. Who can add some arrows to my picture so that it is still a function?

Invite the students to come, one at a time, to the board and to draw yellow arrows so that the resulting yellow relation is still a function. Each time a new arrow is added to the picture, let the class decide whether the relation is still a function. If an incorrect arrow is drawn, erase it and let the student try again.

After about 15 arrows have been drawn in the picture on the board, you should begin to draw new arrows and loops yourself. Do this so that you sometimes add an arrow or loop which makes the relation no longer a function. Each time you add an arrow or loop, ask "Is it still a function?" If "no", ask "Which dot is guilty?" Then erase that arrow and go on.

Most classes become extremely animated over this activity, and even the slowest students should be able to respond, to recognize that an arrow is correct or not, and to point out guilty dots. You can stop this activity whenever you feel the students have tired of it, or you have examined most of the different situations, or you have the same number of arrows as dots and someone comments that any new arrow will make it no longer a function.

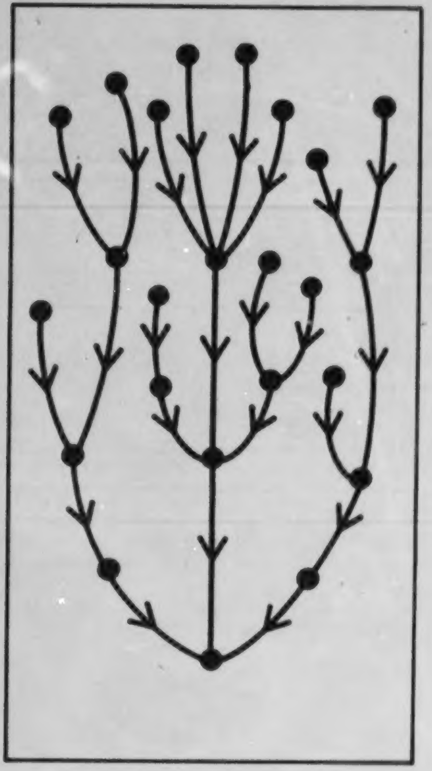
Distribute Worksheet L06 and allow the class about five minutes to work individually. Then collectively discuss that they should have found exactly nine guilty points.

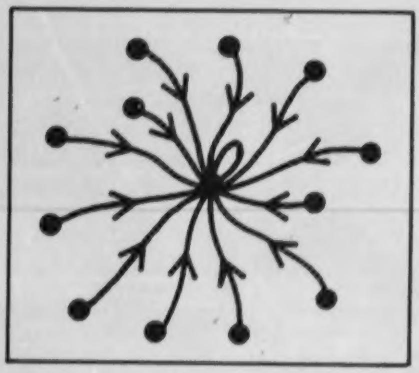
4 L

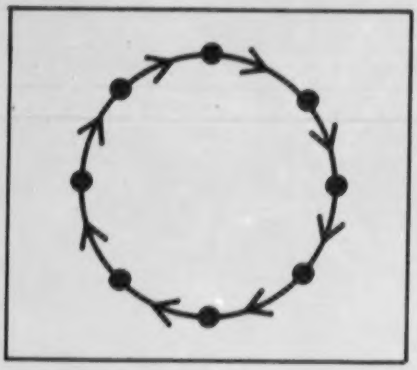
NOTE: As a class project you might invite students to draw artistic pictures of functions for a "Function Art Gallery" to be displayed on a bulletin board.

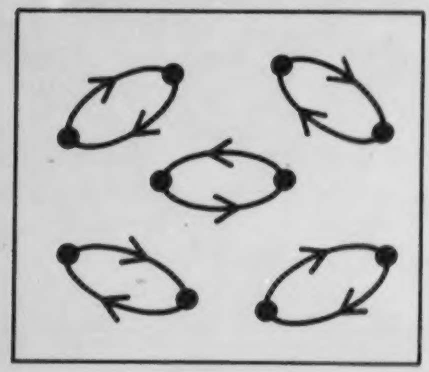
Some interesting functions that could be displayed are shown below.

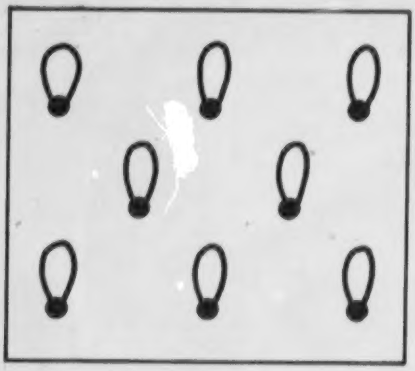
FUNCTIONS











The "Function Art Gallery" consists of abstract art, that is, functions with no specific meanings. In the next part of the lesson, we become more concrete and consider specific functions, where the dots are for people or for numbers.

Exercise 3

T: Is the relation "you are my mother" a function?

S: Yes. I can't have more than one mother.

T: How about the relation "you are my father"? Is it a function?

S: Yes.

T: What about "you are my brother"? Is it a function?

S: No. Some people have more than one brother.

T: Is there anyone in the class with more than one brother? (To each student with more than one brother, say "You are a guilty one".) So "you are my brother" is not a function.

T: Is the relation "you are my grandfather" a function?

S: No. I have two grandfathers.

T: What about "you are my maternal grandfather"?

S: Yes. I have only one maternal grandfather.

T: Suppose we let the dots be for numbers. What do you think about the relation +5? Is it a function?

S: Yes. For each number there is only one number that is five more.

Continue questioning with the relations −5, ×5, $\frac{1}{2}$x. Each is a function.

T: Is the relation "you are more than me" a function?

S: No. For each number there are many larger numbers.

T: Consider the relation "you are a positive divisor of me". Is it a function?

S: No.

T: Why? How many positive divisors does 6 have, for example? (four) Is (+5) • (+3) a function?

S: Yes. It is +8 and +8 is a function.

T: Suppose we have two functions: a yellow function Y and a blue function B. For example, suppose Y is ×3 and B is ×4. Then what is the red relation Y • B?

As you ask this, start making a table on the board.

Y	B	Y∘B
×3	×4	×12

And when you get the response, ×12, write this in the table.

T: Is ×12 a function? (Yes)

T: We also said that (+5) ∘ (+3) is a function.

Add these functions to your table.

T: If we let Y be the function "you are my mother" and B be the function "you are my father" what is the relation Y ∘ B? ("You are my maternal grandfather") Is this also a function? (Yes)

Add this new information to your table.

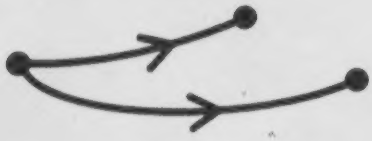
Y	B	Y ∘ B
×3	×4	×12
+5	+3	+8
you are my mother	you are my father	you are my maternal grandfather

T: Suppose we let Y and B be any functions. What can we then say about the red relation Y ∘ B? Is it also a function? What do you think?

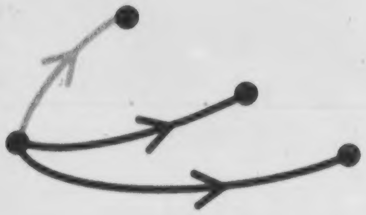
The students will probably say they think Y ∘ B is also a function. Let them try to explain why this is the case. The discussion might need to be helped by questions such as:

T: Suppose I say that Y ∘ B is not a function. I believe there could be a guilty dot where two red Y ∘ B-arrows start. Can you explain why I am wrong?

Let the students give you as much of the explanation as they can. Suggest, if necessary, examining a guilty dot that you believed there would be. Draw two red arrows from a guilty dot.

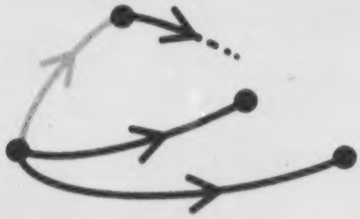


T: Where must a yellow arrow start? (At the dot where a red arrow starts.)



How many yellow arrows can start at this dot? (Only one) Why? (Because Y is a function)

Where must there be a blue arrow? (From the dot where the yellow arrow ends to a dot where a red arrow ends)



55

How many blue arrows can start at the blue dot where the yellow arrow ends? (Only one) Why? (Because B is a function)

This one blue arrow must end at a dot where a red arrow ends. Why? (Because the rule is "yellow followed by blue is red") Can there be two dots where red arrows end if there is only one blue arrow? Can I be right? (No)

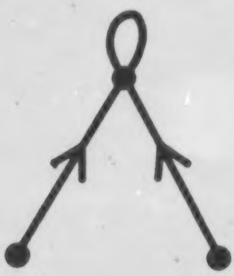
So, $Y \circ B$ is a function, because it has no guilty dots.

If the above discussion does not seem to be fruitful, do not insist on it. Merely state that for any functions Y and B, $Y \circ B$ is also a function.

Exercise 4

If there is still time, give each student a blank sheet of paper. Ask them to draw three dots on their papers and then to draw some function with these dots. Encourage them to try to think of a function that others in the class will not think about. Allow a few minutes thinking time.

T: Now, I will draw my function on the board.



Is it okay?

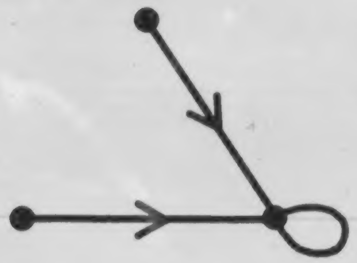
56

Let the students discuss your drawing and agree that it is for a function.

T: Can someone draw a different picture for a function with three dots?

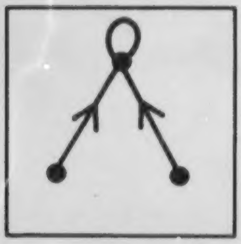
Encourage the students to draw as many different pictures as they can find. A complete set of such pictures is on the following page.

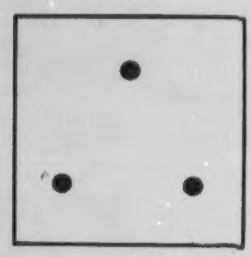
If some student draws this picture,

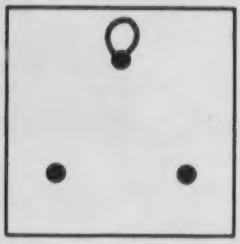


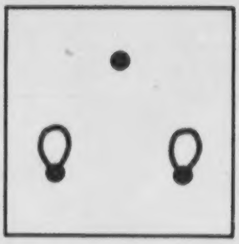
indicate that this is really just the same as your picture. The only difference is that the dot with the loop is in a different position. Agree that these are the same picture and you would like a different picture.

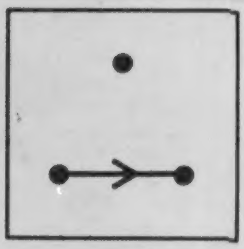
4 L

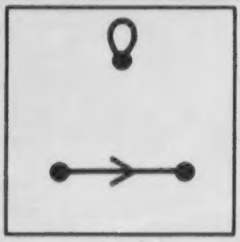


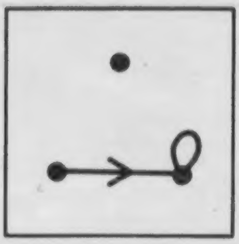


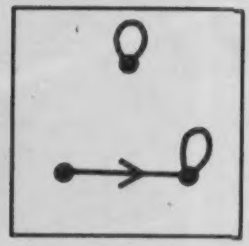


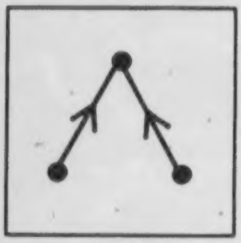


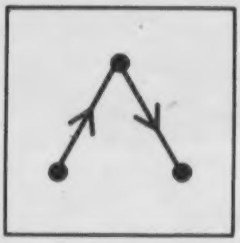


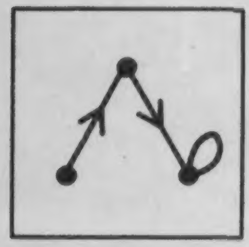


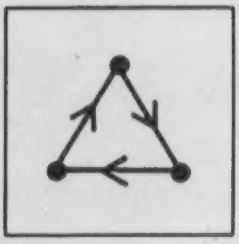


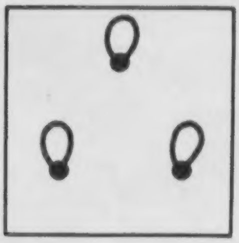


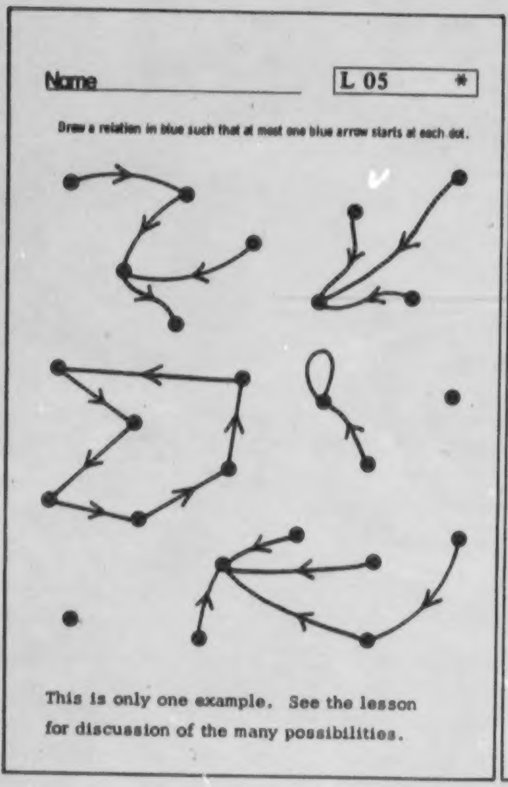








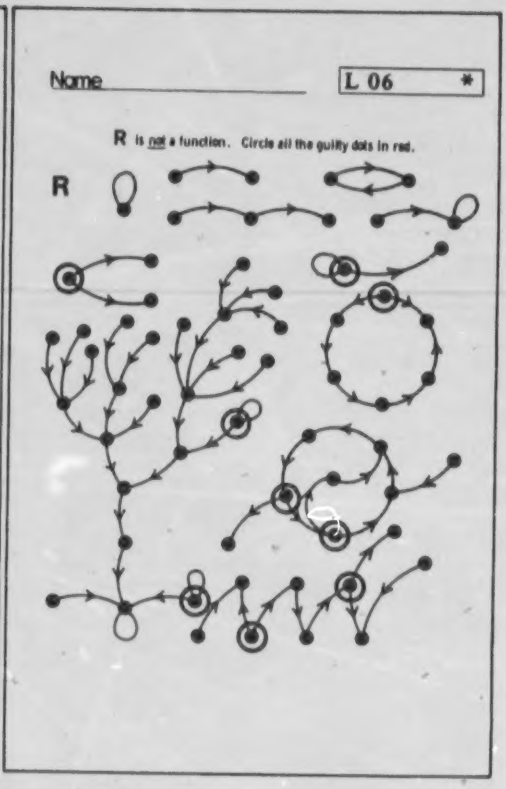




Name ____________________ L 05 *

Draw a relation in blue such that at most one blue arrow starts at each dot.

This is only one example. See the lesson for discussion of the many possibilities.



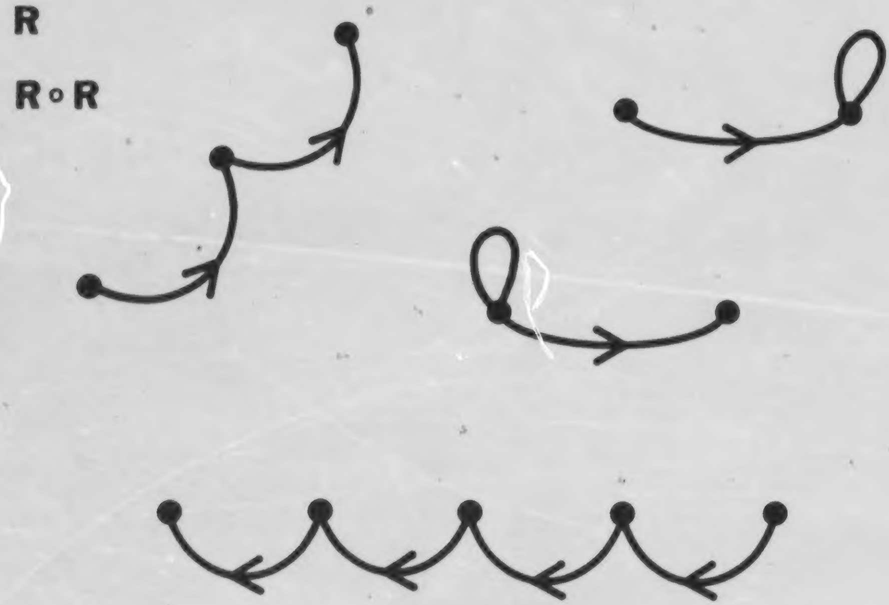
Name ____________________ L 06 *

R is not a function. Circle all the guilty dots in red.

1.3 With and Against

NOTE: The words "with" and "against" in the title of this lesson refer to the direction of arrows for a relation. "With" means "in the direction of the arrow" and indicates using the relation itself in a composite. "Against" means "in the opposite direction of the arrow" and indicates using the opposite relation, or converse of the relation, in a composite. The use of these words in describing a composition of relations should become evident from the illustrations of this lesson.

This lesson is based on the three Worksheets L07, L08, and L09. Draw this portion of Worksheet L07 on the board before the lessons begins.

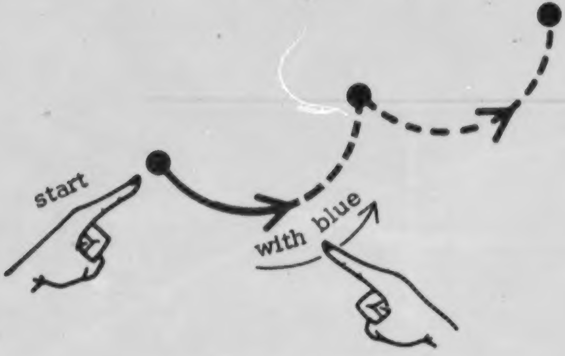


T: The blue relation is called "R".

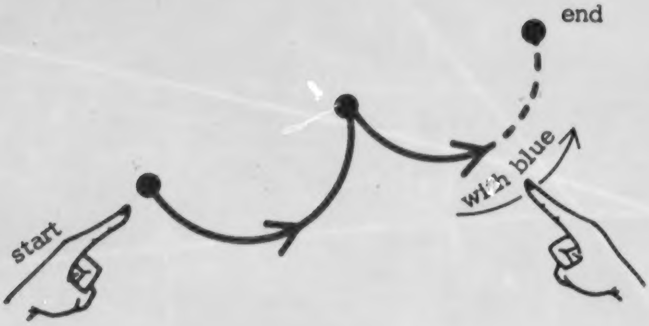
6 L

In the upper left portion of the picture, go through the customary motions (described again in the following commentary), this time drawing red arrows for the relation R • R. At this stage the instructions can be shortened considerably. We do so by saying simply "with blue, with blue" as we trace a blue arrow followed by a blue arrow, and "then red" as we trace the composite red arrow.

T:	With a first blue arrow...	Place your left forefinger at starting dot and trace the first blue arrow with your right forefinger.

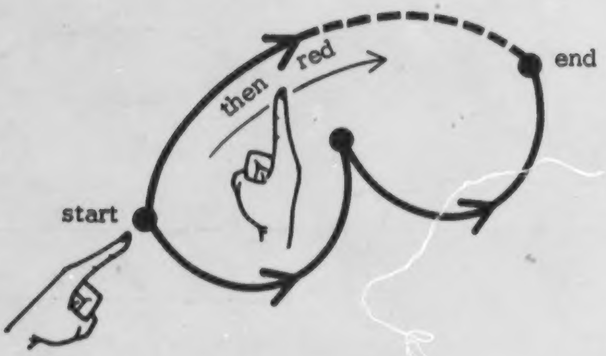


T:	...then with a second blue arrow...	Continue tracing the second blue arrow with your right forefinger



T: ...is a red arrow for R • R.

After your right finger arrives at end dot, trace the red arrow from its start to its end.



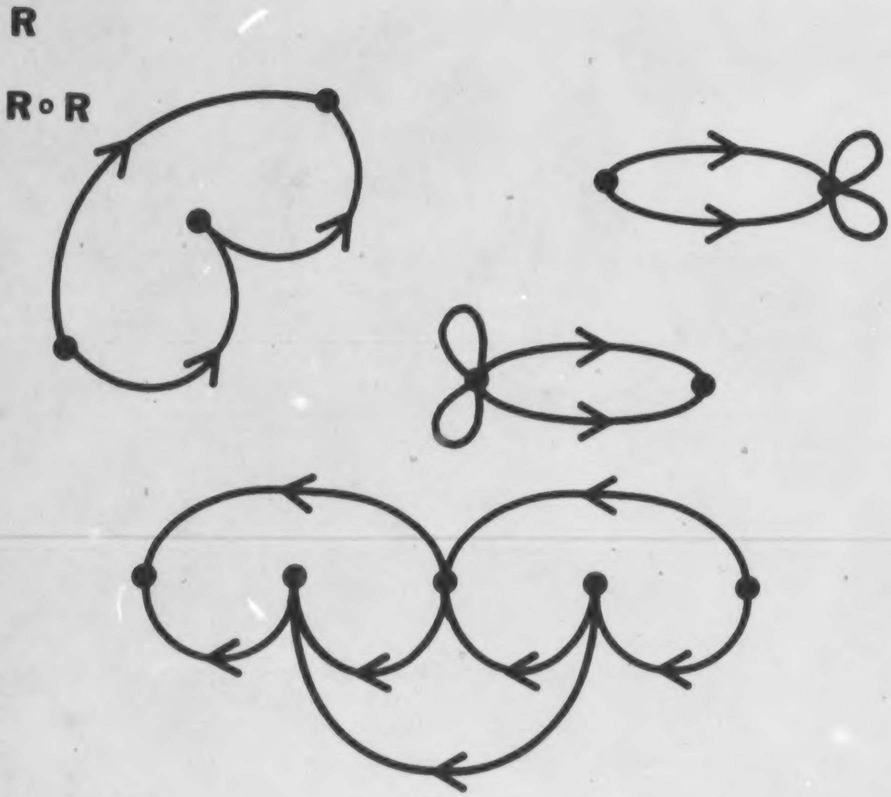
Repeat the tracing motions and say several times:

with blue, with blue; then red.

Of course, these words are short for "with (in the direction of) a first R arrow followed by with a second R arrow gives an R ∘ R arrow".

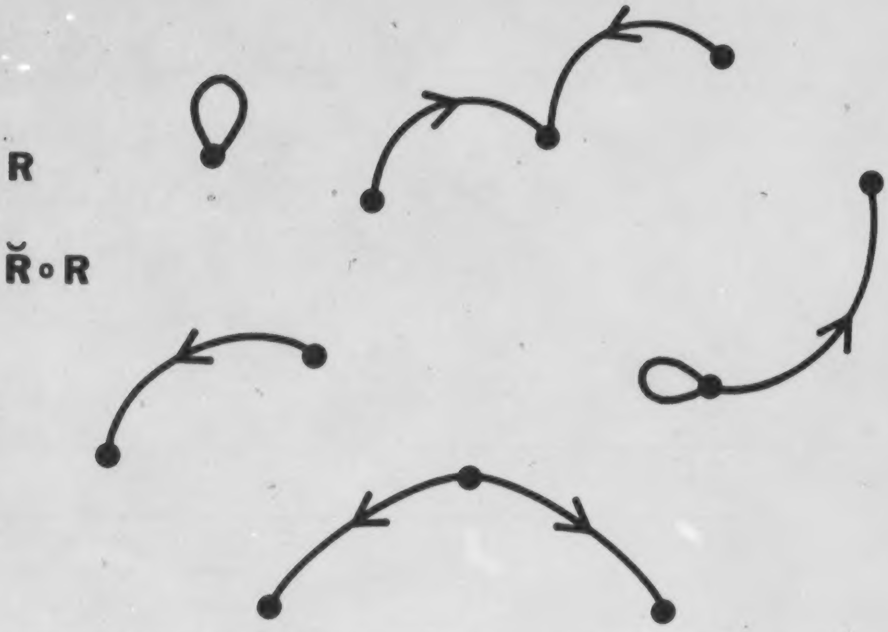
T: Who can show us another red arrow?

Let individual students come to the board and first trace the arrows saying "With blue, with blue; then red", and then draw R ∘ R arrows. The students should quite quickly be able to find the remaining red arrows and loops in your picture.



Distribute Worksheet L07 and have the students work independently drawing all the R ∘ R arrows they can find.

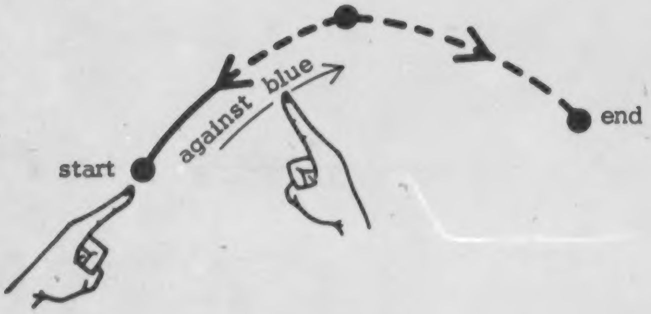
While the students are working, erase the board and draw the portion of Worksheet L08 shown in the next illustration. After several students have finished Worksheet L07, call the class' attention back to the board and to your new picture.



Explain how to draw the arrows of the relation $\breve{R} \bullet R$.

T: Against a first blue arrow...

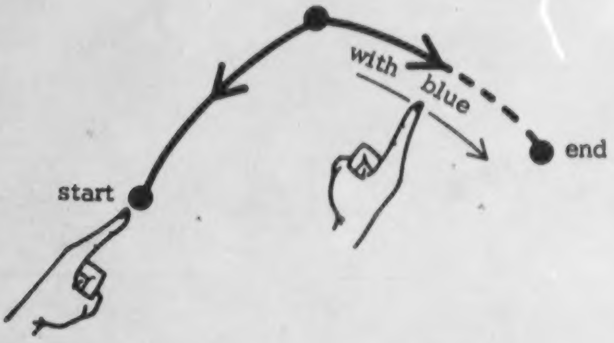
Place your left forefinger on a dot where a blue arrow ends, and with your right forefinger trace against (in opposite direction of) the first blue arrow.



6 L

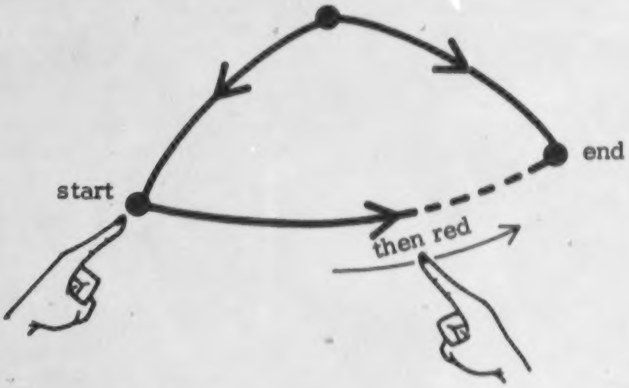
T: ...then with a second blue arrow...

Continue tracing with the right forefinger, this time with the second blue arrow.



T: ...is a red arrow for $\breve{R} \circ R$ (read "$\breve{R} \circ R$" as "against R followed by with R")

After the right forefinger arrives at end dot, trace the red arrow from its start to its end.



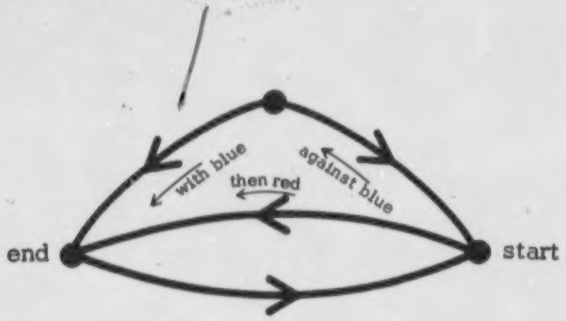
Repeat these motions and the words:

against blue, with blue; then red.

By this we mean "against a first R arrow (i.e., in the opposite direction of a first R arrow) followed by a second R arrow (i.e., in the direction of a second R arrow) gives an $\breve{R} \circ R$ arrow.

T: We could also start here. (Point to where your first red arrow ended.)

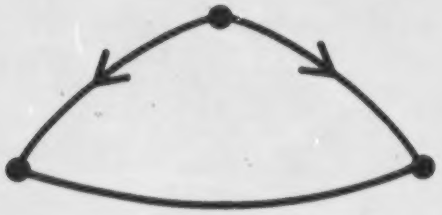
Go through the same motions, using the same words as before, but this time with the starting point shown below.



Again repeat these motions and the words several times:

against blue, with blue; then red.

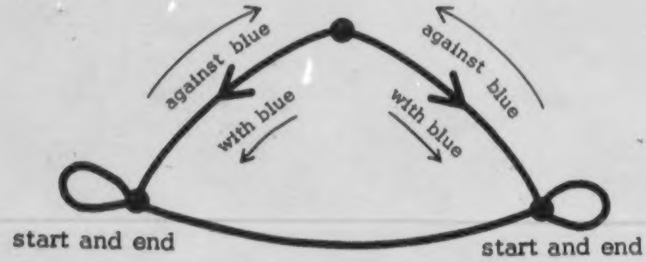
T: With these two red arrows, we have a round trip. Do you remember how we can show these two arrows with one cord? (Erase the red arrows and draw a red cord.)



66

T: We are still not finished here. (First point to the left dot shown below, and then repeat pointing to the right dot.)

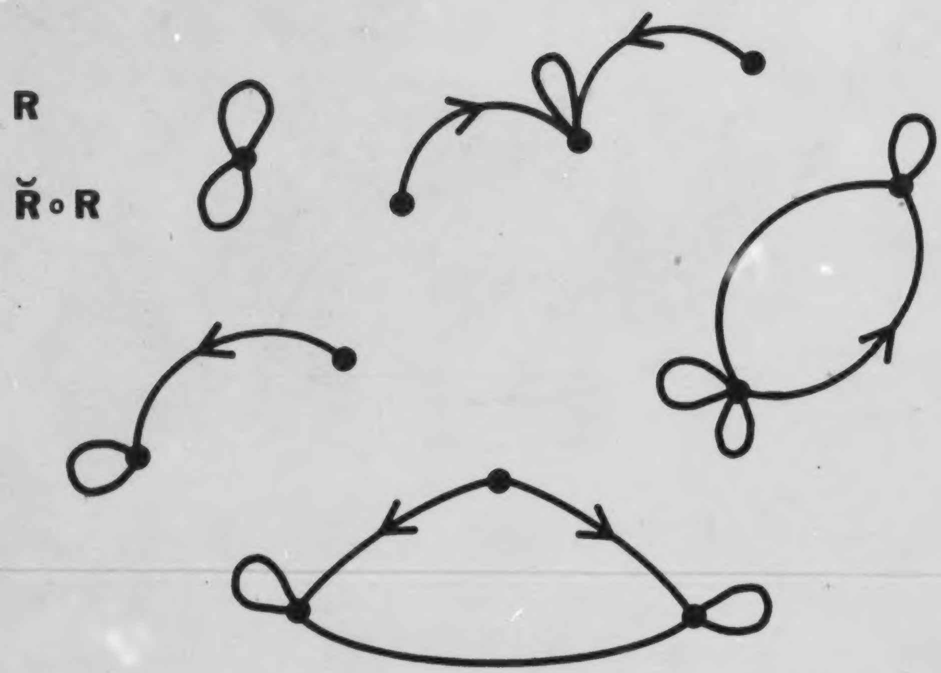
Go through the same motions, saying the same words, but this time starting and ending at each point where a blue arrow ends.



In this case, with your left forefinger at a starting dot, your right finger traces against the R arrow, then with the same R arrow, back to the same dot, giving an $\check{R} \circ R$ loop.

T: Now we know how to draw "against - with" red arrows.

Let individual students trace and draw the other $\check{R} \circ R$ arrows in the picture on the board. Encourage them first to trace the "against blue, with blue; then red" paths before drawing the red arrows. When all the $\check{R} \circ R$ arrows have been found, your picture should look like this.

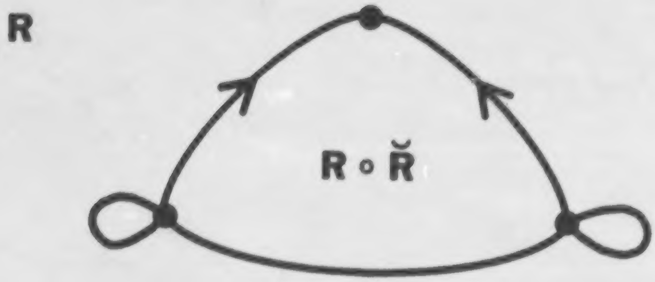


Students will likely comment that there is a red loop at every dot where a blue arrow ends. If not, encourage them to make this observation.

Distribute Worksheet L08 and have the students independently find as many $\breve{R} \circ R$ arrows as they can.

Finally, as students finish Worksheet L08, let them try Worksheets L09 and L10. The code in the center of each worksheet should be enough of a hint for the students to determine the rules for drawing missing arrows, cords, and loops.

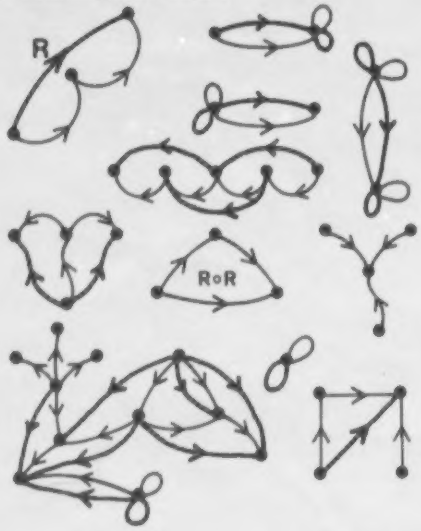
Comment on Worksheet L09:



Notice that there will be an $R \circ \breve{R}$ loop at every dot where an R arrow starts. Also, be aware that some students may have difficulty in tracing "with" and "against" loops. Since a loop has no direction, there is no difference between "with" a loop and "against" a loop.

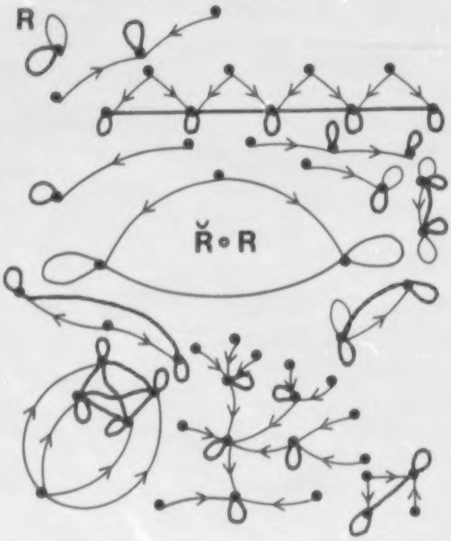
Name ____________________ L 07 *

Draw all the missing red arrows and loops.



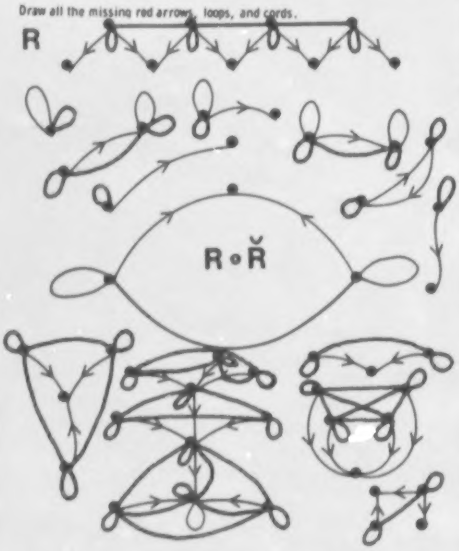
Name ____________________ L 08 **

Draw all the missing red arrows, loops, and cords.



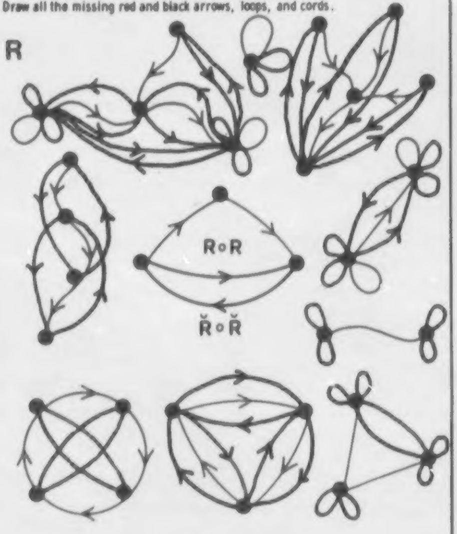
Name ____________________ L 09 **

Draw all the missing red arrows, loops, and cords.



Name ____________________ L 10 ***

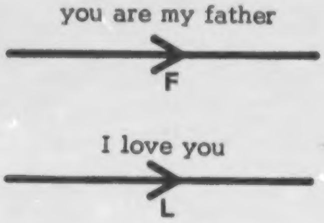
Draw all the missing red and black arrows, loops, and cords.



1.4 Arrow Messages #1

Before the lesson begins, give each student two pieces of paper and three different colored pencils (red, blue, and yellow are used in this lesson description, but may be substituted if necessary).

T: With these two relations (draw the arrow codes shown below in an upper corner of the board) we can send some messages. The blue arrows are for the relation F, "you are my father". The red arrows are for the relation L, "I love you".



T: Is the relation F a function?

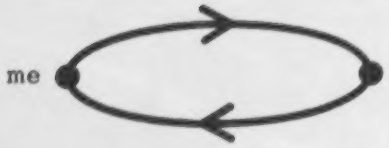
S: Yes; everyone has at most one father.

T: Is the relation L a function?

S: No; I can love lots of people.

T: Now, let's decide what messages we can send with F and L arrows. Think of this dot as being for you, the speaker. (Draw a dot and label it "me". Throughout this lesson, the dot labeled "me" indicates the speaker.)

Draw this arrow picture.



T: What message are you sending?

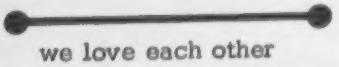
S: "I love you and you love me."

S: "We love each other."

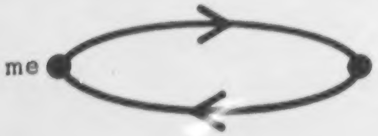
T: Could you draw the same arrow picture in a simpler way?

S: Draw a red cord for the round trip arrows.

Erase the red arrows and draw a red cord.



Next draw this arrow picture.



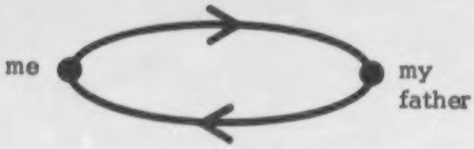
T: What message do you send with this picture?

S: "You are my father and I love you."

S: The same message is "I love my father."

S: That's wrong. I'm saying, "My father loves me."

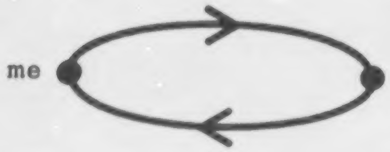
If the students have difficulty deciding, you might indicate on the picture who the other dot represents.



my father loves me

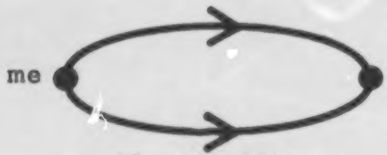
T: Could someone draw an arrow picture for the message, "I love my father."

You may get several incorrect responses before a correct one. For example,



S: It can't be right. The message here is "I am your father and I love you."

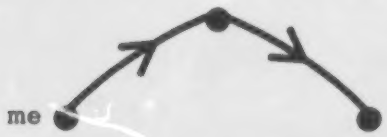
Someone should eventually suggest:



I love my father

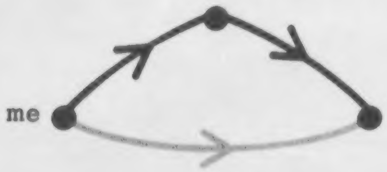
S: That's right. I'm saying, "You are my father and I love you" or "I love my father."

T: Now let's use three dots in a message. (Draw the following picture.) What can we say about this?



S: Someone says to his father, "You are my father" and his father says, "You are my father" to his father.

T: If we draw this yellow arrow, what relation could it be for? (Draw in the yellow arrow.)



9 L

NOTE: The dot at the start of the yellow arrow is the speaker (hence labeled "me") and the dot at the end of the yellow arrow is the person being spoken to.

S: "You are my grandfather."

S: F ∘ F.

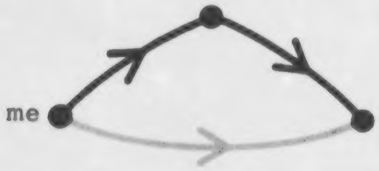
S: "You are my father's father."

T: Do you know another English word to express this?

S: "You are my paternal grandfather."

T: With these three dots, can you send a different message using blue and red arrows?

If the students do not suggest another message, you can draw another; for example,



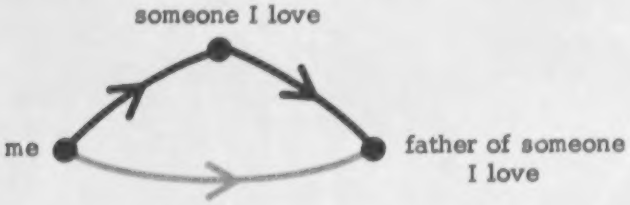
T: What could this yellow arrow be for?

S: "I love you, dad."

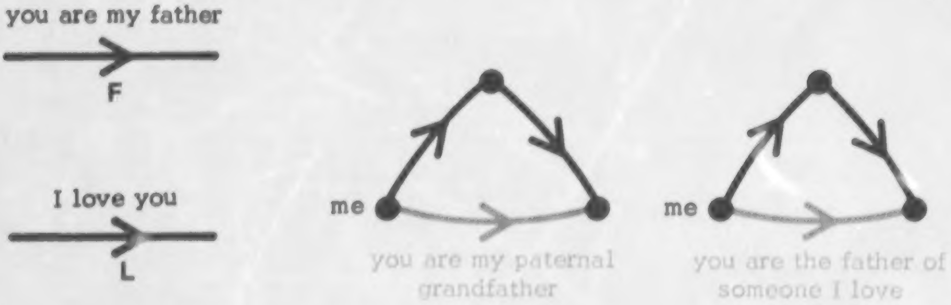
S: No. The blue arrow is for, "You are my father", and the red arrow is for "I love you", but I am not saying "I love you" to my father.

S: It is for, "You are the father of someone I love".

Allow enough discussion for the class to agree on the meaning of the yellow arrow. If necessary, you can identify the other dots.


someone I love
me
father of someone I love

Erase the board except for these pictures.


you are my father
F
I love you
L
me
you are my paternal grandfather
me
you are the father of someone I love

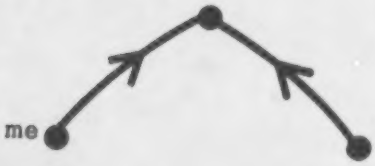
T: We've looked at two different arrow messages with three dots. With just three dots, how many different arrow messages, using these blue and red arrows, do you think there could be?

A student might answer:

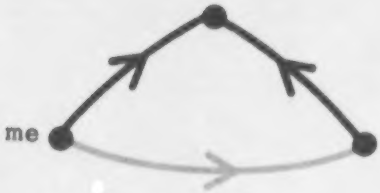
S: There can be three of them.

T: Alright. Please draw the third picture.

For example,



T (drawing the yellow arrow): What could this yellow arrow be for?



S: "You love my father."

T: Are you sure we have all the messages?

S: There are six in all.

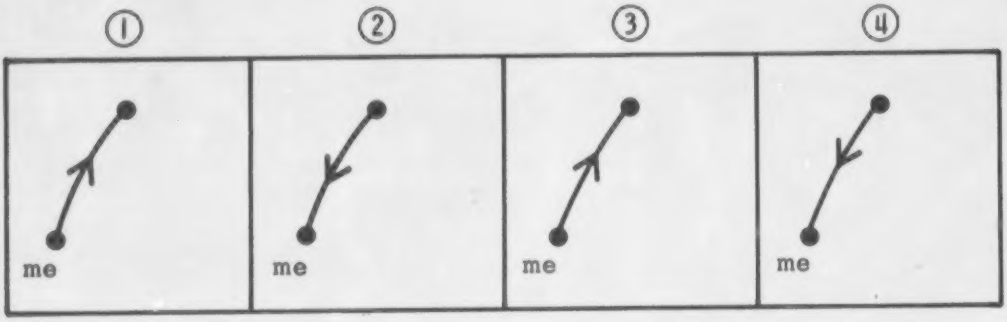
S: Nine.

Let the students make several guesses.

At this stage of the lesson, begin a systematic process for counting the number of different arrow messages.

T: Let's try to find all the messages with just two of the dots. How many are there? (Four)

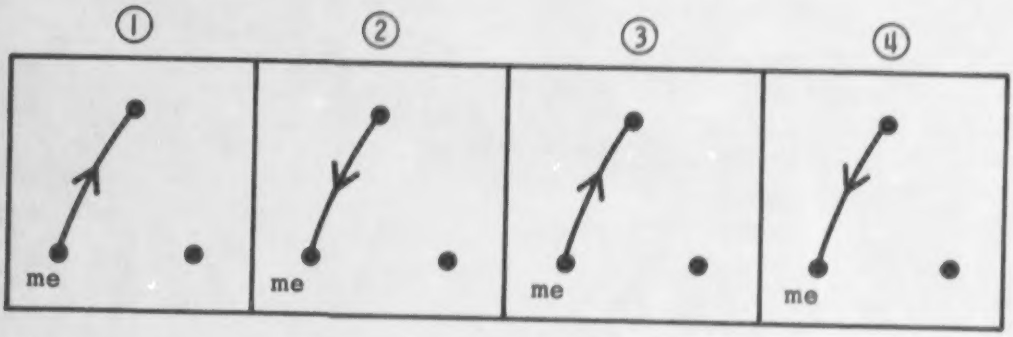
Accept suggestions and draw them as they are given, until you have all four possibilities.



T: Now do we have all the messages that can be drawn with two dots?

S: Yes. There are two colors and each arrow can go two different directions.

T: Now, how many different arrow messages with three dots? (Add a third dot to each of the four pictures.)



Guesses might range from 6 to 18 or 20.

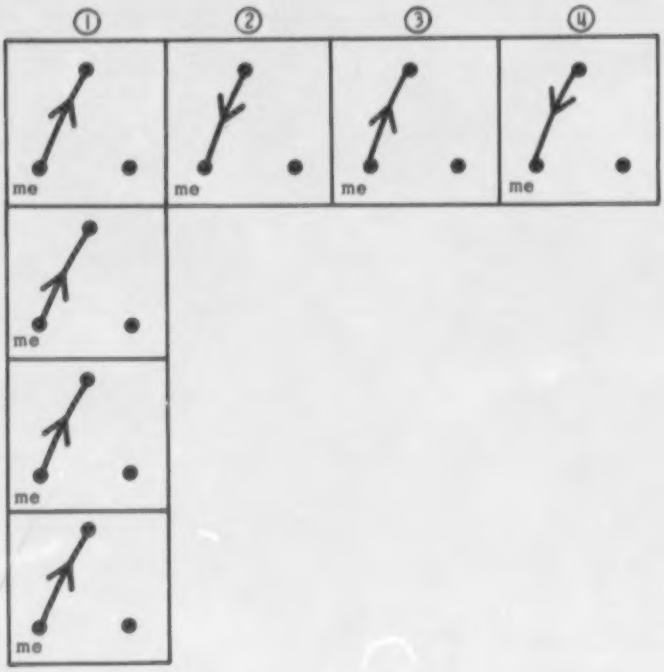
T: How can we find out?

S: Draw them all.

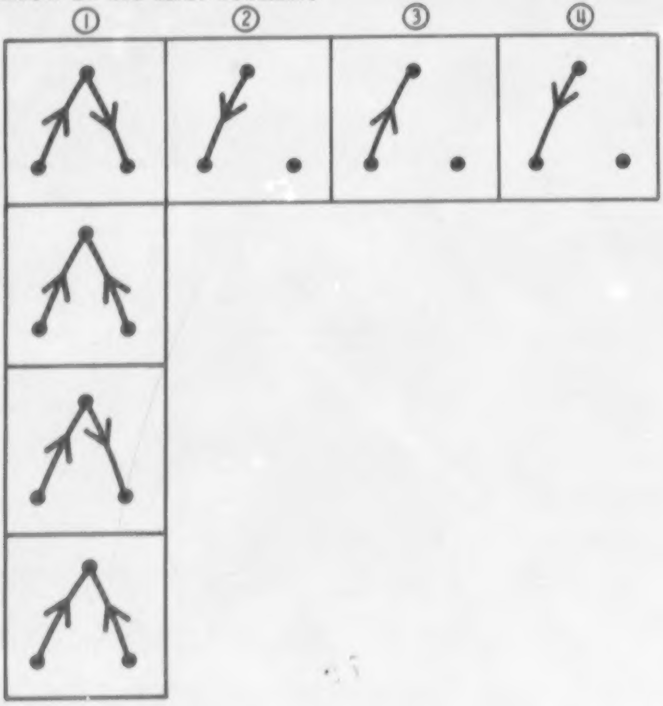
T: Alright, but let's organize our drawings so that we don't get confused.

S: Take each of those four possibilities (pointing to board) with two dots and draw it four times; then draw a second arrow in each.

Follow this suggestion if it is given, or provide it yourself. Repeat picture ① until there are four of them, and so that the board looks like this.


①
②
③
④
me
me
me
me
me
me
me

T: Let's consider the 1st picture. How many ways can I draw the second arrow in the first column?



S: Four ways.

S: I see! There are 16 messages in all.

S: Yes. There will be four columns.

S: Like a table!

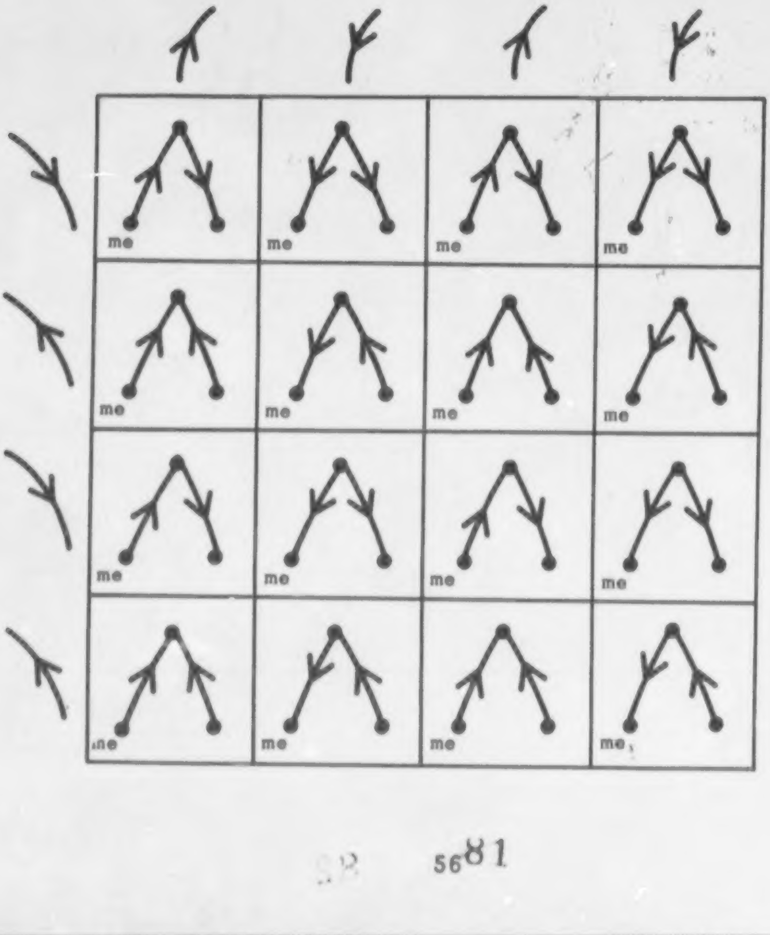
T: Do all of you think there are 16? Why?

Let someone summarize the explanation:

For each of the four ways we can draw an arrow between the first two dots, there are four ways we can draw an arrow between the last two dots.

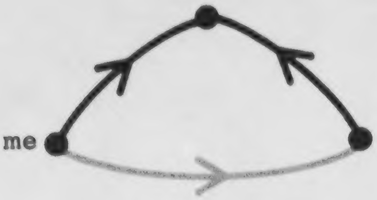
T: I would like each of you to try to finish drawing all the 16 possible arrow pictures.

The following table shows all 16 possibilities.



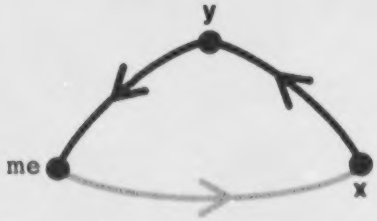
After the students have drawn as many as they can, continue the class discussion.

T: Let's try to figure out some of these messages. For example, this one (second row, third column).



S: "You love my father."

T: And this one (second row, second column).

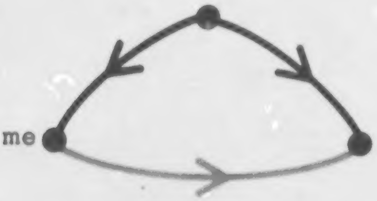


S: "Your love loves me."

To avoid confusion here you might want to label the dots, as above.

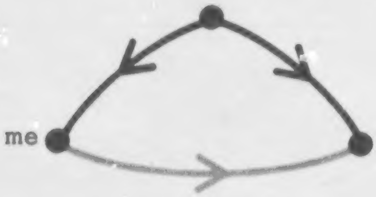
S: x says to y, "I love you" and y says to me, "I love you" so I say "you (x) love (y)" -- "you love someone who loves me."

T: How about this one (third row, second column)?



S: "You are the father of someone who loves me."

T: And this one (third row, fourth column)?



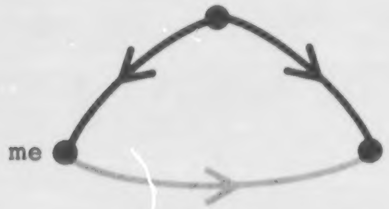
S: It isn't possible! You can't have two fathers.

S: I see a guilty dot.

S: "You are my father " is a function. It can't have a guilty dot. There can't be two arrows starting at one dot.

Cross out this picture in your table.

T: Another (first row, fourth column).



S: "My child loves you."

T: Not quite. You could be a mother and still say that.

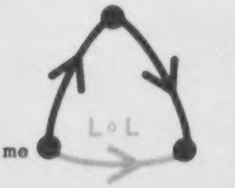
S: "I am the father of someone who loves you."

T: Good.

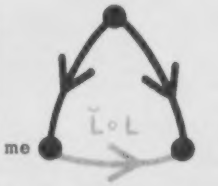
If the class is enjoying this activity you can continue eliciting the messages as long as time permits.

9 L

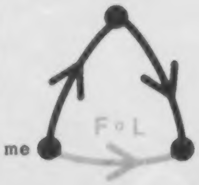
Below is a complete list of the messages.



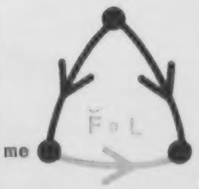
I love a person who loves you.



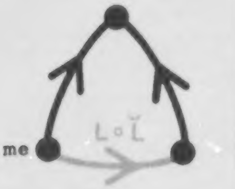
Some person loves both of us.



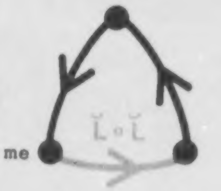
My father loves you.



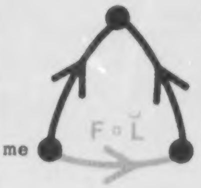
I am the father of a person who loves you.



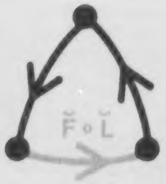
We love the same person.



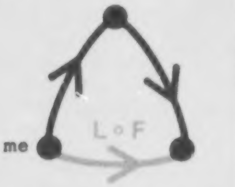
You love a person who loves me.



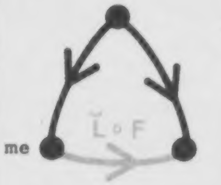
You love my father.



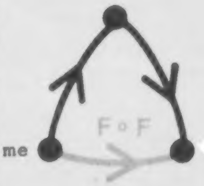
I am the father of someone you love.



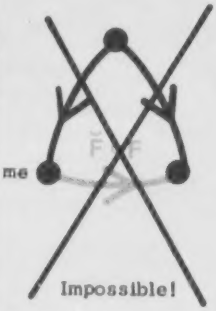
You are the father of someone I love.

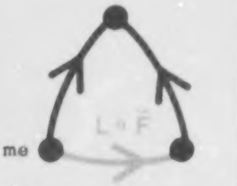


You are the father of someone who loves me.

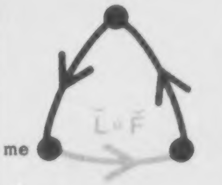


You are my paternal grandfather.

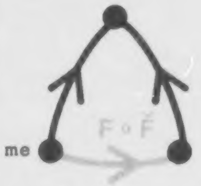

F̆ ∘ F
me
Impossible!



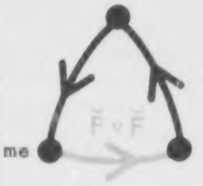
I love your father.



Your father loves me.



We have the same father.



I am your paternal grandfather.

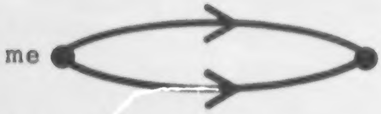
1.5 Arrow Messages #2

This lesson will be based on Worksheets L11, L12, L13, L14, L15, and L16.

Begin by distributing Worksheets L11 and L12, and ask the class to look at the first picture at the top of L11.

T: What is the blue arrow for? What is the red arrow for?

As you accept answers to these questions, copy the arrow picture on the board.



T: What message do I send with this arrow picture? (Or, if this is not understood, you might ask, "What information or message does this picture give?")

S: "You are my father and I love you."

S: "I love you, father."

S: "I love my father."

T: Good. Now write your message below the picture and try to find the other messages.

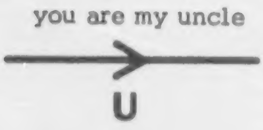
As you walk around the room helping individual students, do not object to any way of expressing the messages unless they are clearly wrong. For example, the second message on L11 could be "you love me, daddy" or "you are my father and you love me", but clearly could not be "I am your father and you love me" nor "I love my child".

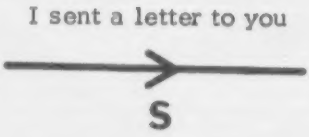
If necessary, remind students that the bottom picture on L11 has a red cord, and ask what the cord means.

On Worksheet L12 you might accept messages involving "grandfather", although strictly speaking each involves "paternal grandfather" or "father's father".

When several of the students have finished L12, ask the class to put the worksheets aside and continue with the following collective lesson.

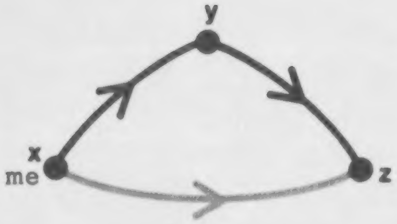
Erase what is on the board and draw these arrows.





T: We can send new messages with these two relations. U is the blue relation, "you are my uncle" and S is the red relation, "I sent a letter to you".

Start drawing this picture,



and as you draw the yellow arrow, ask:

T: What could the yellow arrow be for?

S: For the relation U ∘ S.

T: Yes, but in words?

S: "You sent a letter to my uncle."

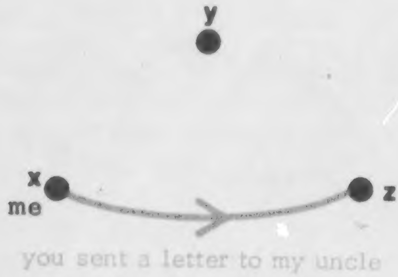
S: No, if I am x, then my uncle is y, and you (z) did not send a letter to y. If you are z, then my uncle sent you a letter.

T: So could the yellow arrow be for the relation, "My uncle sent a letter to you"?

Let the class discuss this until they agree that this message is correct. Then write the message below the yellow arrow.

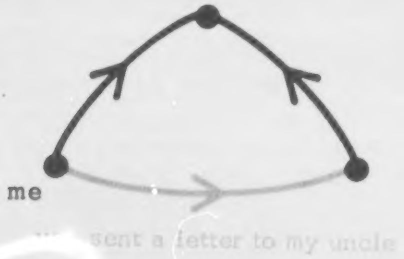
12L

Draw this picture on the board.


y
x
me
z
you sent a letter to my uncle

T: Someone thought the previous message was, "You sent a letter to my uncle." Can someone draw an arrow picture with red and blue arrows so that the yellow arrow from x to z is for the relation, "you sent a letter to my uncle".

If there are guesses, analyze each one to see why it is or is not reasonable. If there are no responses, ask some leading questions such as "Who is my uncle?", i.e., "Who is x's uncle?" (y); "To whom did you send a letter?", i.e., "To whom did z send a letter?" (y) The class should finally agree on the arrow picture below.


me
sent a letter to my uncle

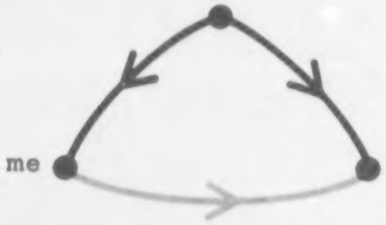
Let the students compare this arrow picture with the preceding one and notice how they differ.

If the class seems to be capable of generating more messages, continue in this vein.

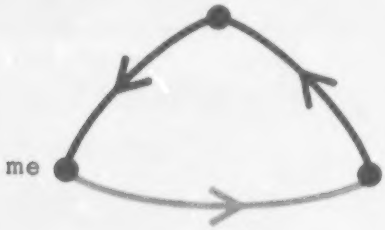
T: Can someone give us another message in English that we can show with an arrow picture?

You must be careful at this point to accept only those messages that result from the composite of two of the relations U, S, $\breve{U}$, $\breve{S}$. (Recall that "$\breve{U}$" means "the converse of U" or "against U", and "$\breve{S}$" means "the converse of S" or "against S". See Section 1.3.) There are 16 such possibilities, all of which can be obtained from the table at the end of Section 1.4 Arrow Messages #1 by replacing "you are my father" with "you are my uncle" and "I love you" with "I sent a letter to you".

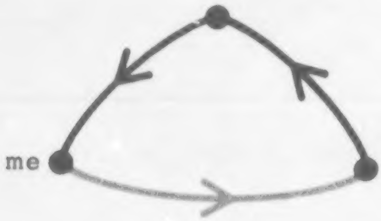
For example, a student might suggest, "you are an uncle of someone who sent a letter to me",



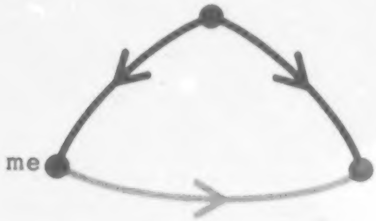
or "one of your uncles sent a letter to me",



or "I am an uncle of someone to whom you sent a letter",

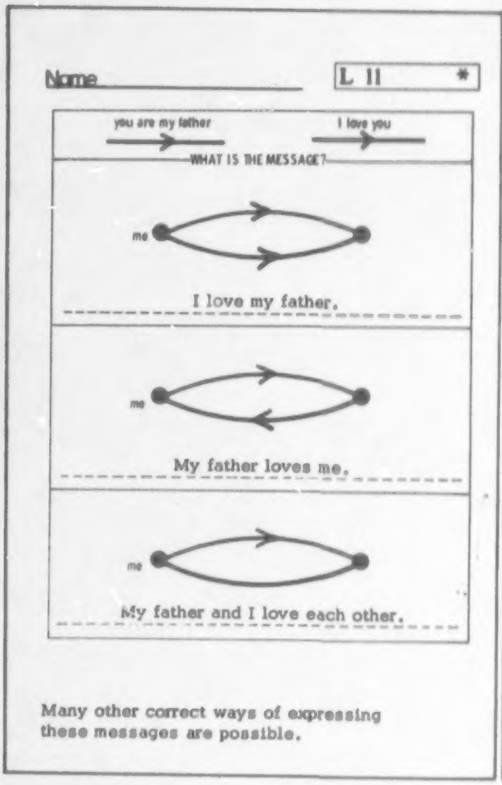


or "someone sent a letter to you and a letter to me".



If no reasonable English messages are suggested, you can suggest any or all of the above and help members of the class to draw the corresponding composite yellow arrow. For variety, you could ask students to suggest different arrow pictures with red or blue arrows, draw yellow composite arrows and let other students find the corresponding English messages.

Worksheets L13, L14, L15, and L16 are available for the rest of the lesson.



Name

L 11 *

you are my father

I love you

WHAT IS THE MESSAGE?

me

I love my father.

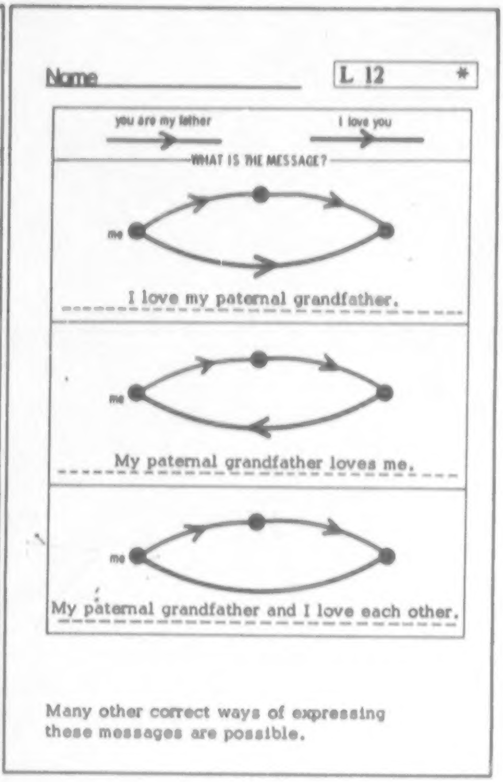
me

My father loves me.

me

My father and I love each other.

Many other correct ways of expressing these messages are possible.



Name

L 12 *

you are my father

I love you

WHAT IS THE MESSAGE?

me

I love my paternal grandfather.

me

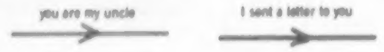
My paternal grandfather loves me.

me

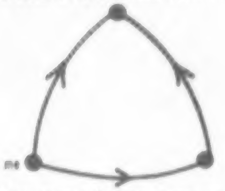
My paternal grandfather and I love each other.

Many other correct ways of expressing these messages are possible.

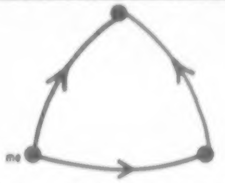
Name ____________________ L 13 *


you are my uncle
I sent a letter to you

WHAT COULD THE BLACK ARROW BE FOR ?



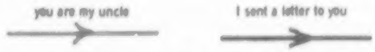
You sent a letter to my uncle.



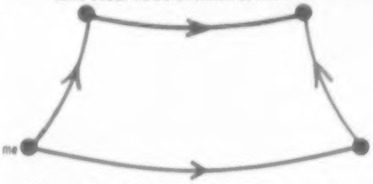
I sent a letter to your uncle.

There are other correct ways of describing the black relations.

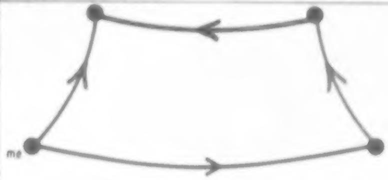
Name ____________________ L 14 **


you are my uncle
I sent a letter to you

WHAT COULD THE BLACK ARROW BE FOR ?



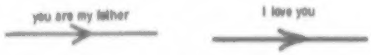
My uncle sent a letter to your uncle.



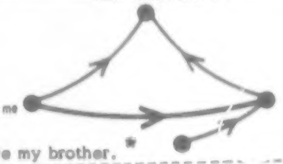
Your uncle sent a letter to my uncle.

There are other correct ways of describing the black relations.

Name ____________________ L 15 ***

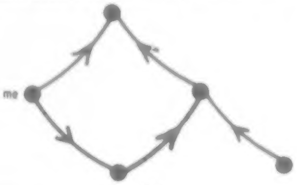

you are my father
I love you

WHAT IS THE MESSAGE ?



I love my brother. *

WHAT COULD THE BLACK ARROW BE FOR ?

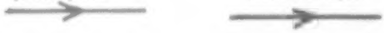


You love my brother. **

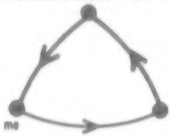
* Other correct ways of expressing this message are possible.

** There are other correct ways of describing the black relations.

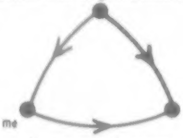
Name ____________________ L 16 ****


you are my uncle
I sent a letter to you

Draw blue and red arrows so that the black arrow is for the given relation.



you sent a letter to a person who sent a letter to me



I am an uncle of a person who sent a letter to you

Make up your own message and draw an arrow picture for it.

Each student's message and corresponding arrow picture needs to be checked individually.

2

STRING GAMES and LOGICAL THINKING

The language of strings is used as a vehicle to present various situations involving sets which stimulate logical thinking. The situations presented in the lessons of this chapter are similar to those experienced earlier in the CSMP program, and special attention is given to situations involving the already very familiar String Games. Continuing the experiences in the L strand of CSMP Mathematics for the Intermediate Grades, Part I, explicit attention is paid to some precise terms such as: at least; at most; exactly; none of; each of; etc. The String Game situations involve the students in thinking critically about the relations "is a divisor of", "is a multiple of", "is larger than", etc., as well as challenge them to think logically about the relationships between various sets and to give persuasive arguments. The situations are rich, they encourage deductive reasoning on the part of students, and they draw upon the students' numerical and nonnumerical knowledge with the aid of string pictures.

CAPSULE LESSON SUMMARIES

Section 2.1: Play the String Game with A-Blocks and with numbers. Review the rules of the game and present some challenging situation in which to discuss "not" labels.

Section 2.2: One at a time correctly put some numbers (game pieces) in the string picture for a String Game. Analyze the information obtained from each of these clues and determine the string labels. Play the Numerical String Game.

Section 2.3: Using string pictures, determine whether a given statement is true, false, or you can't tell. Consider statements involving "exactly", "at most", and "at least". Write true statements, false statements, and statements about which you can't tell for a given string picture. Given a statement, make changes in a string picture so that the statement is true (false).

Section 2.4: Using correct plays in a String Game as clues, see how quickly you can identify the string cards while playing the game. Play the Numerical String Game.
(Worksheets L17, L18, L19, L20)

Section 2.5: Analyze a situation where the starting clues of a String Game determine the strings. Play a type of collective String Game where the object of the game is to make as few plays as possible to determine the strings. Play the Numerical String Game.

Section 2.6: Continue with exercises similar to those of Section 2.3. Using intermediate situations from Numerical String Games, consider which numbers can definitely be put in the picture and which are still indefinite.

Section 2.7: Repeat the exercises of Section 2.5 with different starting situations. Play the Numerical String Game with new scoring rules.

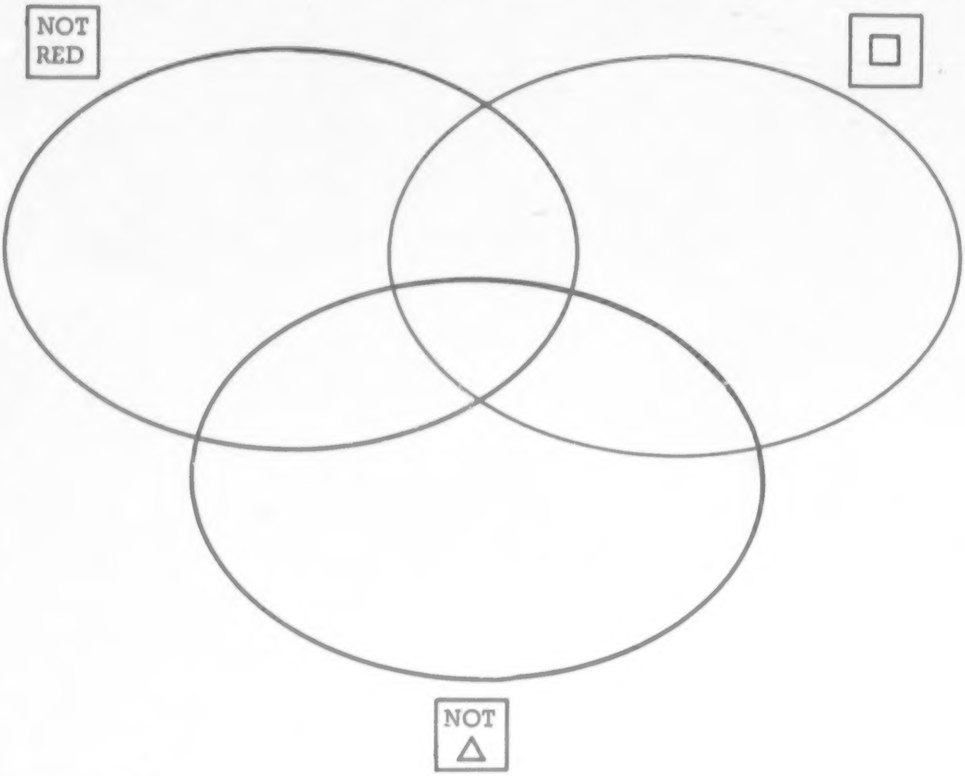
Section 2.8: Extend the experiences of Section 2.7 to include hatching in string pictures as an additional clue. Begin to discuss the strategy of certain plays in a String Game.
(Worksheets L21, L22)

2.1 The String Game[*]

Your class has played many versions of the String Game, some with shapes (A-Blocks) and some with numbers. We suggest that you begin this lesson by briefly reviewing the rules of the game (if necessary) and establishing "penalties" for infringement on the silence rule. This review should set the stage for the many times during this semester that the String Game will be played.

Exercise 1

Play the String Game, Version B: The String Game with A-Blocks, using "Not" Cards. An interesting game could be played using three strings; for example,



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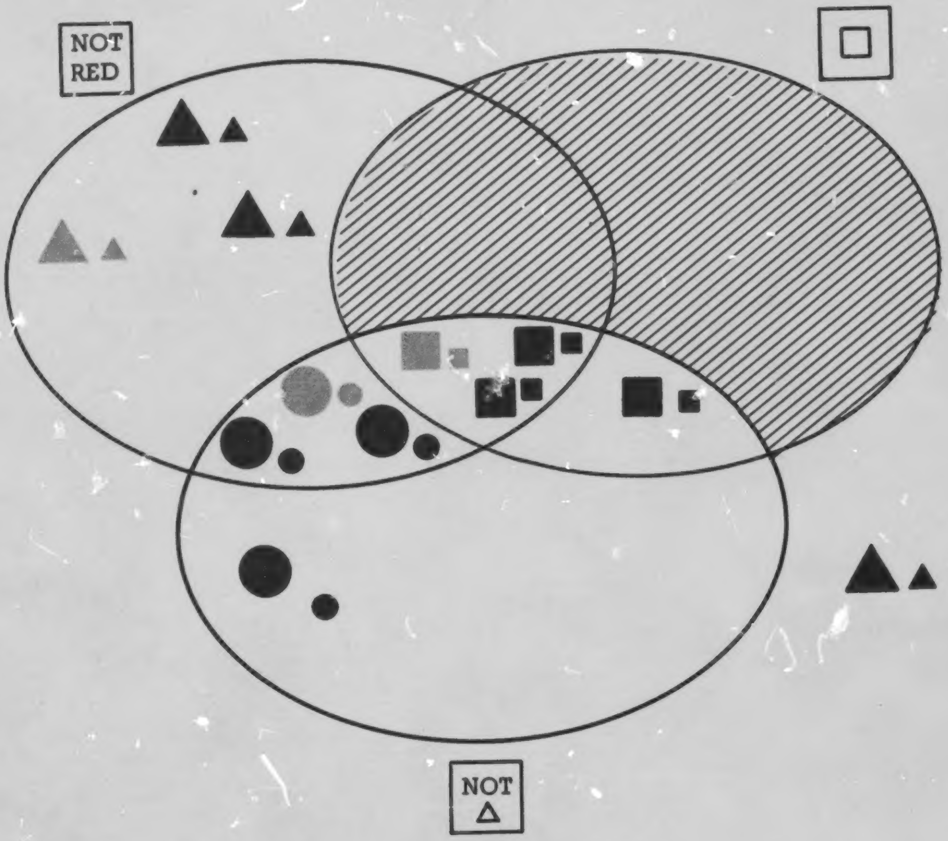
[*] See the appendix to this chapter: The String Game, for a description of the different versions of the game mentioned in this lesson.

This would be a challenging game for the class, and some interesting discussion about "not" descriptors could follow the game. If your class is not ready for such a difficult game, use just two strings at this time.

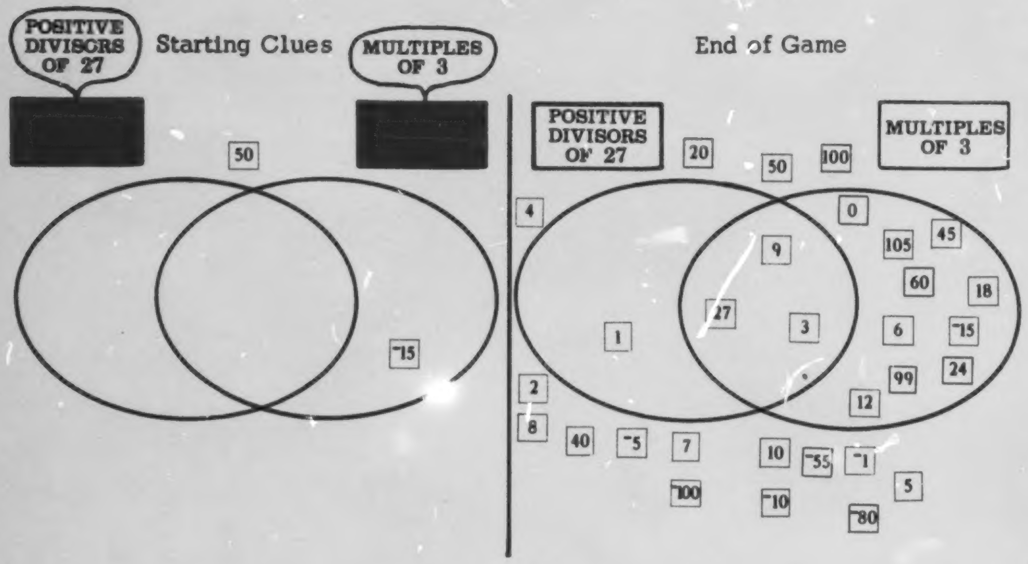
You should be aware that this game will be difficult to judge without a "crib-sheet", so we suggest you refer to the picture below during the game. This picture shows where all the pieces should be at the end of the game.



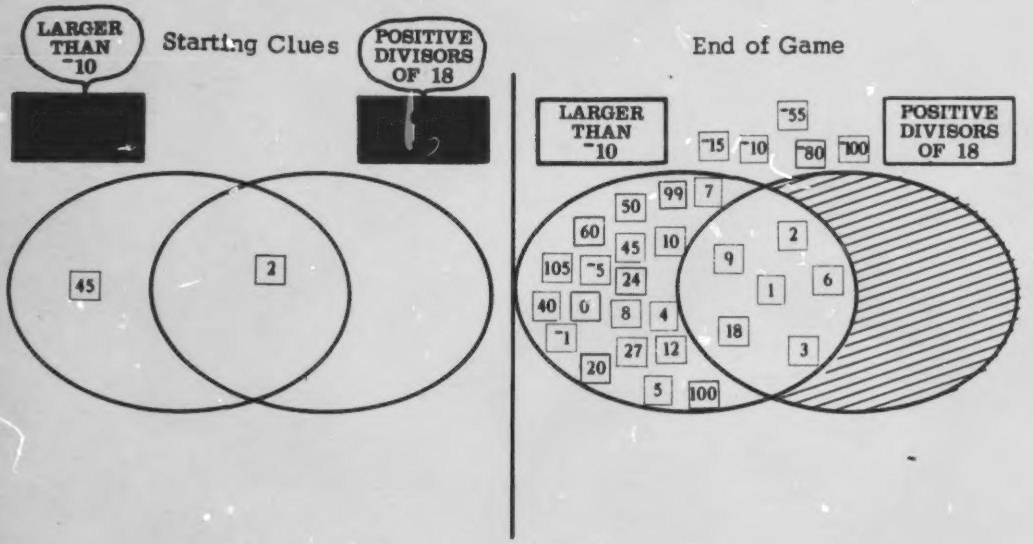
Exercise 2

Play the String Game, Version D: The String Game with Numbers: Multiples, Divisors, and Order, once or twice as time and interest permit. Two games are suggested here, one with no empty regions and one with an empty region.

You should, however, decide for yourself what kind of game would be most appropriate for your class. If you have little time remaining, you can shorten the time needed to play a game by giving more starting clues.


POSITIVE DIVISORS OF 27
Starting Clues
MULTIPLES OF 3
50
⁻15
End of Game
POSITIVE DIVISORS OF 27
20
50
100
MULTIPLES OF 3
4
0
105
45
9
60
18
27
3
1
6
⁻15
99
24
2
12
8
40
⁻5
7
10
⁻55
⁻1
5
⁻100
⁻10
⁻80

No Empty Regions

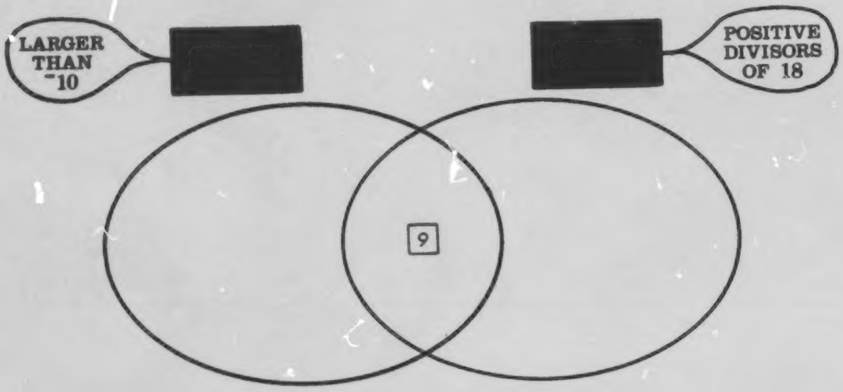

LARGER THAN ⁻10
Starting Clues
POSITIVE DIVISORS OF 18
45
2
End of Game
LARGER THAN ⁻10
⁻55
⁻15
⁻10
⁻80
⁻100
POSITIVE DIVISORS OF 18
99
7
50
60
45
10
2
9
105
⁻5
24
6
1
40
0
8
4
18
⁻1
27
12
3
20
5
100

One Empty Region

2.2 String Game Analysis #1

Exercise 1

Before the lesson begins, set up your board for the String Game as illustrated below. The bubbles show what is on the hidden tags. Tape two demonstration analysis sheets (included in the Numerical String Game Kit) on the board; one to the left of the red string and one to the right of the blue string.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: We are going to play the string game today, but first we will look at what information we get from some plays. This can help us be better players. When we play the game, it is like solving a detective story. We want to find out what the red and blue strings are for, and each piece that is played is a clue. Let's play a game together.

Direct the class' attention to the board.

T: Here is the list of possibilities for the red string (point to the analysis sheet at the left) and here is the list for the blue string (point to the analysis sheet at the right). Our first clue is that 9 is in the center region; 9 is inside both strings. What information does this clue give us about the strings?

S: The red string cannot be for the multiples of 2, because 9 is not a multiple of 2.

T: Could the blue string be for the multiples of 2?

S: No, because 9 is also inside the blue string.

Use a crayon or heavy felt-tip pen to cross out this possibility from both the RED and BLUE lists.

RED

~~MULTIPLES OF 2~~	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Let the students eliminate as many labels as they can from both the RED and BLUE lists. Each time they cross out a label on one list because the corresponding string cannot have that label, they should see that the same label should be crossed out on the other list. A number in the center region gives the same information about both strings. Do not allow a label to be crossed out until an adequate explanation has been given. For example,

S: The red (blue) string cannot be for the positive divisors of 12, because 9 is not a positive divisor of 12.

When all the information from this clue has been discussed, your class should find there are only six remaining possibilities for each string.

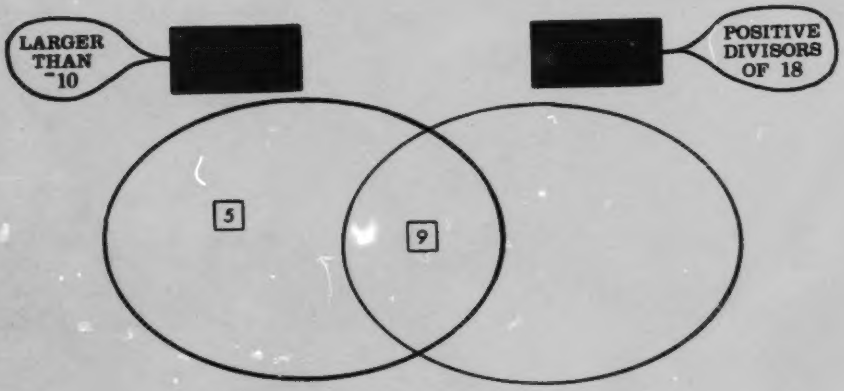
RED

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
SMALLER THAN 50	~~SMALLER THAN 10~~	ODD NUMBERS	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
SMALLER THAN 50	~~SMALLER THAN 10~~	ODD NUMBERS	~~POSITIVE PRIME NUMBERS~~

T: Now, I'll give you another clue. 5 is in the red string but not in the blue string. (Put "$\boxed{5}$" on the board.) What new information does this clue give us?



S: The red string cannot be for the multiples of 3, because 5 is not a multiple of 3.

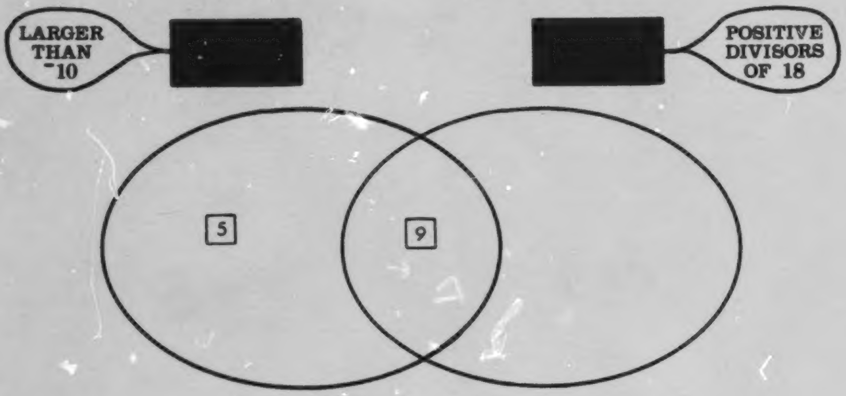
T: Could the blue string be for the multiples of 3? (Yes)

If it is not suggested by a student, you should lead the class to realize that with this clue whatever remaining label the red string cannot have, the blue string can have, and vice versa. For example,

S: The blue string cannot be for the odd numbers, because 5 is an odd number and is not in the blue string.

T: Could the red string be for the odd numbers? (Yes)

When all the information from this clue has been discussed, there should be only three remaining possibilities on each list.



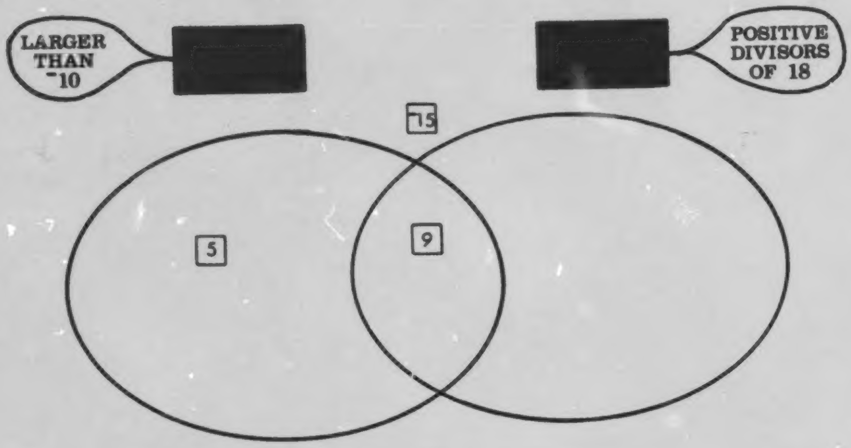
RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ⁻10
SMALLER THAN 50	~~SMALLER THAN ⁻10~~	ODD NUMBERS	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: My third clue is that $^{-}15$ is in the outside region. (Put " $\boxed{^{-}15}$ " on the board.) What new information does this clue give us?

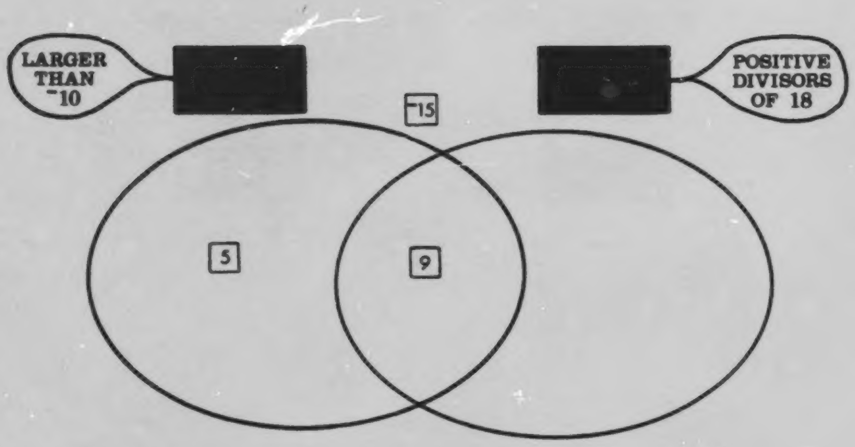


Let the students eliminate labels from both the RED and BLUE lists. For example,

S: The red string cannot be for the odd numbers, because $^{-}15$ is odd and is not inside the red string.

S: The blue string cannot be for the multiples of 3, because $^{-}15$ is a multiple of 3 and is not inside the blue string.

The class should find that these three clues determine the red string, and that there are two possibilities remaining for the blue string.



RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

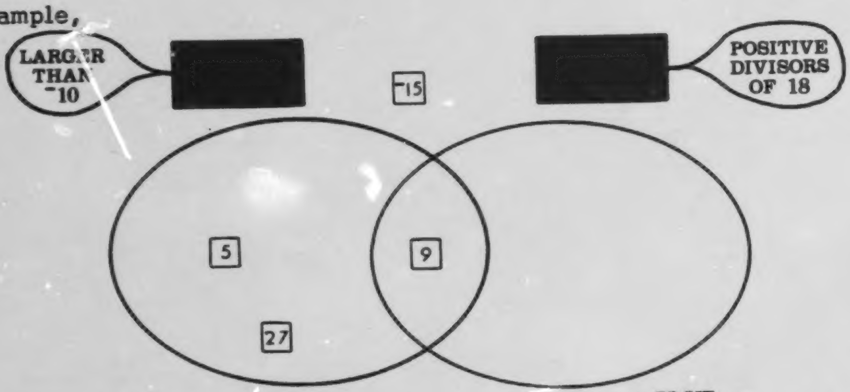
BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: Now we know that the red string is for the numbers larger than $^{-}10$. There are still two possibilities for the blue string. We need another clue. Who can suggest a number to play? I'll put it in the correct place. Try to choose a number so that when we know where it is in the picture, we can determine the blue string.

This problem is difficult and you should allow the students to make several trials, if necessary. If they choose any of the numbers $^{-}100$, $^{-}80$, $^{-}55$, $^{-}10$, $^{-}5$, $^{-}1$, 0, 1, 3, 4, 7, 8, 10, 12, 20, 24, 40, 45, 50, 60, 99, 100, or 105, then ask a student where to put it.[*] Since it is already determined where these numbers are in the picture, playing one of them gives no new information. If they choose 2, 6, 18, or 27, then the play will determine the blue string. You may have to suggest one of these numbers yourself, if, after several trials, no student suggests one of them.

For example,


LARGER THAN ¯10
POSITIVE DIVISORS OF 18
¯15
5
9
27

RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ¯10
~~SMALLER THAN 50~~	~~SMALLER THAN ¯10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	~~LARGER THAN ¯10~~
~~SMALLER THAN 50~~	~~SMALLER THAN ¯10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

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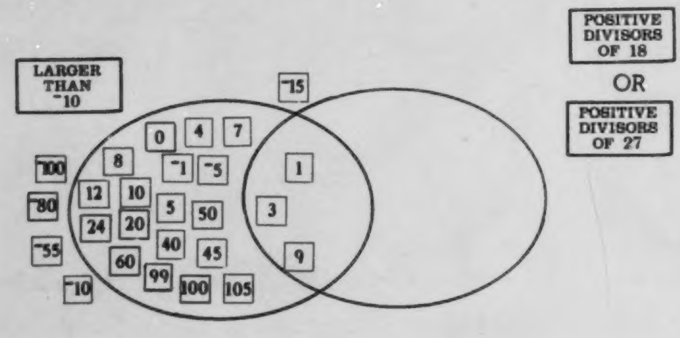
[*] The picture below shows where to put these numbers in the picture. At this point we know that the red string is for the numbers larger than ¯10, and that the blue string is either for the positive divisors of 18 or of 27.


LARGER THAN ¯10
¯15
POSITIVE DIVISORS OF 18
OR
POSITIVE DIVISORS OF 27
0
4
7
8
¯1
¯5
1
¯100
12
10
¯80
5
50
3
24
20
¯55
40
45
9
60
99
¯10
100
105

105

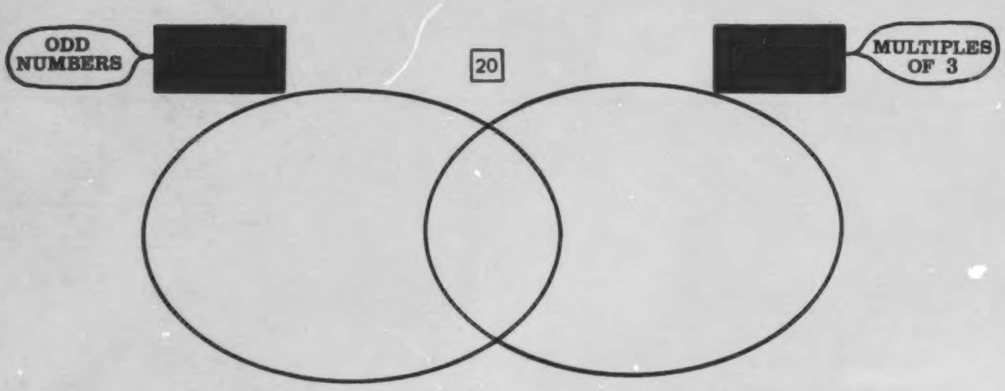
S: The blue string cannot be for the positive divisors of 27, because 27 is not inside the blue string. So, the blue string must be for the positive divisors of 18.

Show the class the hidden tags.

Exercise 2*

Repeat Exercise 1, moving a little more quickly perhaps by encouraging one student to eliminate several similar labels simultaneously, using the following situation and sequence of clues.

Starting situation and first clue:



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

---

* If Exercise 1 took a considerable amount of your class period, you may wish to skip this exercise and go on to Exercise 3, playing the String Game in the time that remains.

Crossed out labels from first clue:

RED

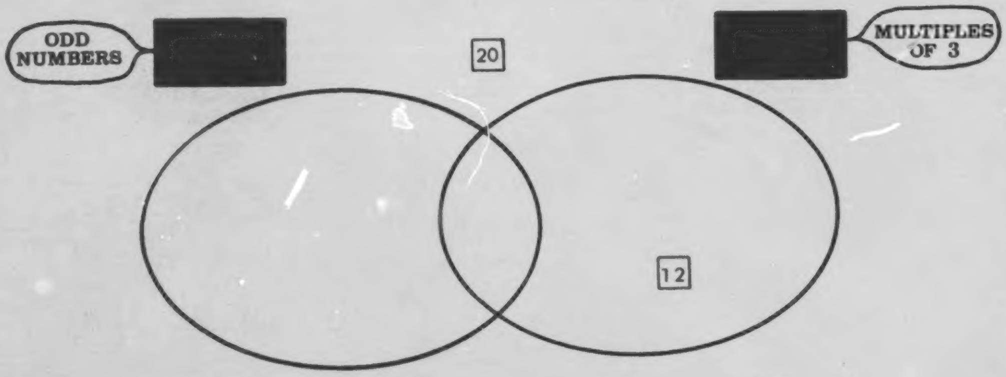
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

NOTE: The same labels are crossed out on both lists, because this clue gives the same information about both strings; 20 is outside of both strings.

Second clue:



RED

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

3L

Crossed out labels from second clue:

RED

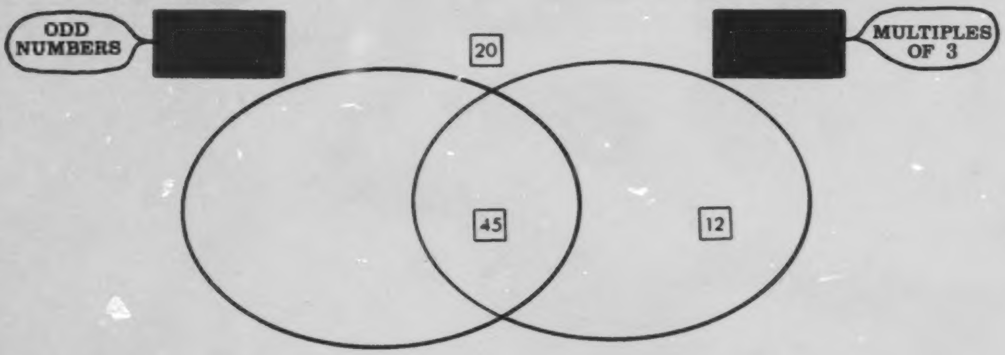
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27	LARGER THAN 50	~~LARGER THAN $^-$10~~
~~SMALLER THAN 50~~	SMALLER THAN $^-$10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^-$10~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^-$10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

NOTE: Of the possibilities remaining after the first clue, those that get crossed out on the RED list do not get crossed out on the BLUE list, and vice versa.

Third clue and crossed out labels from this clue:



RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^-$10~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^-$10~~	[✓ ODD NUMBERS]	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	[✓ MULTIPLES OF 3]	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^-$10~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^-$10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

NOTE: These three clues determine both the red string and the blue string.

Exercise 3

Play Version D (variation #1 with two strings) of the String Game with numbers. Give each student a list of the string labels from the individual String Game Analysis Pad.

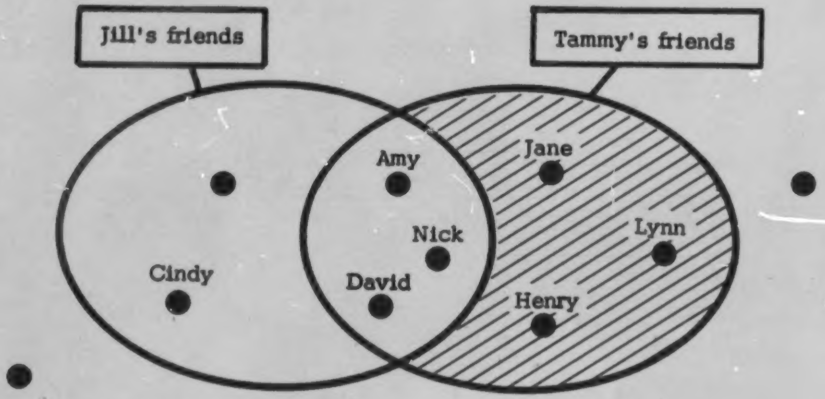
T: Now we are going to play a usual String Game. Each of you has a list of the possible labels for the red string and the blue string. You should try to use this list during the game to help you discover what is on the hidden tags. Try to cross out some of the possibilities as numbers get put in the picture.

In order to give the students time to do some of their own analysis between plays, you should pause briefly between turns when a player has placed a piece correctly. Such pauses in the game should encourage students to use the clues in crossing out labels on their individual lists.

2.3 Logical Thinking #1

Exercise 1

Draw this string picture on the board.


Jill's friends
Tammy's friends
Amy
Jane
Nick
Lynn
Cindy
David
Henry

Ask the students to make any observations or comments they wish about the picture. You will likely receive a variety of comments, for example,

S: Amy is both Jill's friend and Tammy's friend.

S: Jill and Tammy share three friends.

S: Tammy has exactly three friends that are not friends of Jill.

S: Some dots do not have names.

S: Jill has at least five friends.

Encourage your students to make accurate statements, and let the class discuss statements about which there is some disagreement.

T: Okay, now I am going to make some statements and you tell me whether what I say is true or false or you can't tell. You should explain your answers. My first statement is:

Jill has exactly five friends.

S: False. More dots can be put in the red string.

S: Yes, but we can't tell. More dots could be in the red string, but we don't know if there are any more for sure.

T: Very good. You can't tell about this statement. Could you change my statement so that it is true?

S: Jill has at least five friends.

Some students might suggest "Jill has at most five friends" about which you can't tell. You may have to remind the class of the meanings of "at least" and "at most".

NOTE: The phrases "at most" and "at least" are very difficult. Gradually your students will master them by experience and exposure to many different contexts in which they are used. Don't worry if some students are still having problems with this language; have patience and continually remind them of the meanings of these phrases. They will encounter this language regularly throughout the CSMP Intermediate Grades Program and eventually it will become a familiar part of their vocabulary.

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In the following dialogue, we have listed on the left a variety of other statements you could use for this exercise. On the right we give a correct response. Your students may at first make quick incorrect responses; encourage them to think before responding and to discuss each other's answers.

T: Jill has at least four friends.	S: True. Jill has more than four friends.
Cindy is a friend of both Jill and Tammy.	False. Cindy is a friend of Jill but not a friend of Tammy.
Jill has at most one friend that is not a friend of Tammy.	False. There are two dots for friends of Jill that are not friends of Tammy.
Jill and Tammy have at most three friends in common.	Can't tell. They could have three or more than three friends in common. More dots could be in the middle, but we are not sure.

Here would be another good place to ask students to change the statement so that it is true. Encourage students to discuss this situation and the meanings of any suggested statements. A similar true statement is:

Jill and Tammy have at least three friends in common.

T: Tammy has exactly three friends that are not friends of Jill.	S: True. The hatching means there are exactly three dots in that part of the picture.
Brad is a friend of Tammy but not a friend of Jill.	False. Only Jane, Henry and Lynn are friends of Tammy but not friends of Jill.

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T: Brad is a friend of Jill but not a friend of Tammy.	S: Can't tell. Brad could be, but we don't know for sure where he is in the picture.

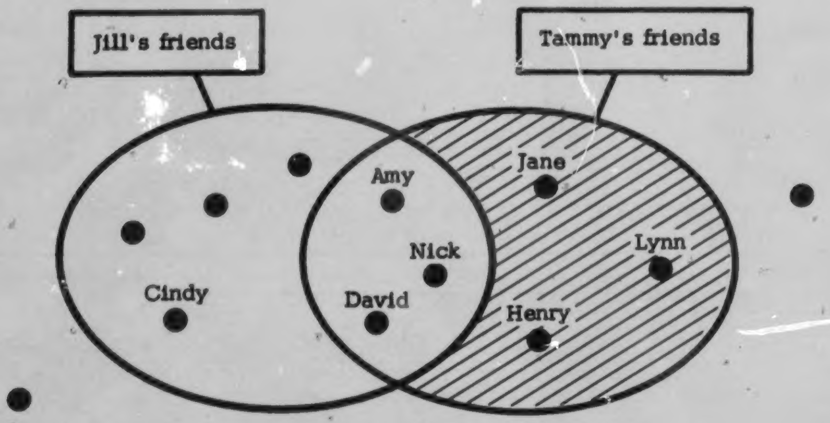
Encourage students to discuss the possibilities for where Brad is in the picture.

You could change the exercise here and ask students for some statements of particular types. Be specific. Ask for a true statement, for a false statement, and for a "can't tell" statement. For any suggested statements, ask the rest of the class for agreement that it is a true, false, or a can't tell statement. Then continue.

T: Jill has more friends than Tammy.	S: Can't tell. Jill could have more friends, but we are not sure.

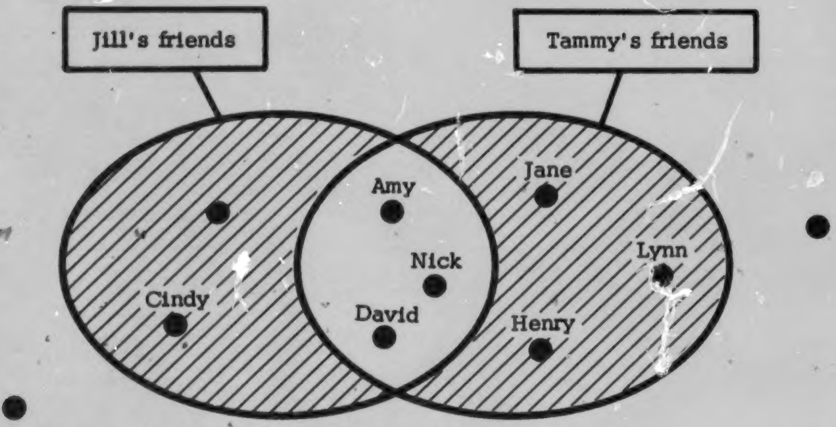
Ask students to change the picture so we are sure this last statement is true.

Let the students experiment and discuss the problem. They should discover that more dots must be added to the left part inside the red string. For example,



Discuss why we are now sure that Jill has more friends than Tammy. (Any other friends of Tammy must be in the center and so also are friends of Jill.) Some students might want to do some more hatching. This would be okay but not necessary.

You could also discuss whether the picture could be changed so that we are sure that Tammy has more friends than Jill. One way to do this would be to hatch the left part inside the red string with just two dots in it.



Return to the original picture.

T: Now I will tell you a true statement. Sam is a friend of Tammy. Can you draw a dot for Sam and label it?

Let a student locate Sam in the middle.

T: What else do we know about Sam?

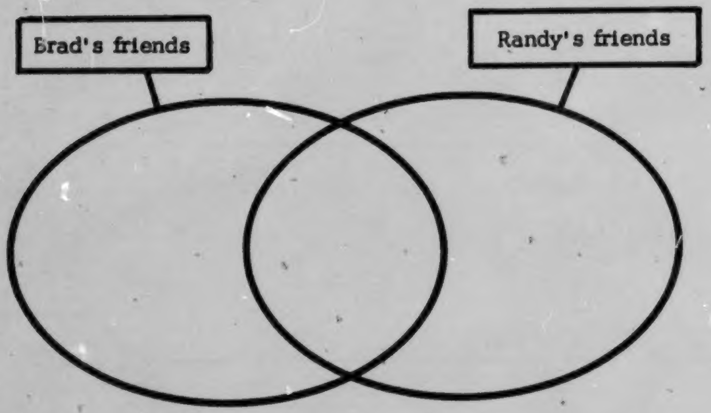
S: Sam is also a friend of Jill.

T: Here is another true statement. Sally is a friend of Jill. Who can locate Sally in our picture?

Let the students discuss the situation and decide that they can't be sure where to locate Sally. She could be a friend of both Jill and Tammy, hence in the middle. Or she could be a friend of Jill but not a friend of Tammy and hence in the left part inside the red string.

Exercise 2

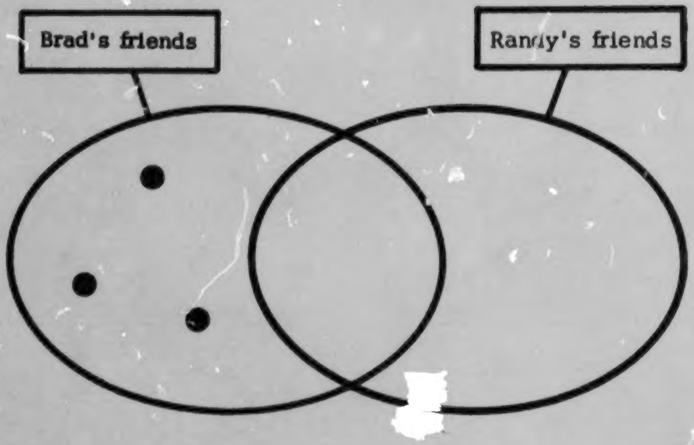
Erase the board and draw this picture.



T: I will tell you some things about Brad's and Randy's friends. You have to help me fix the picture so it is correct. First:

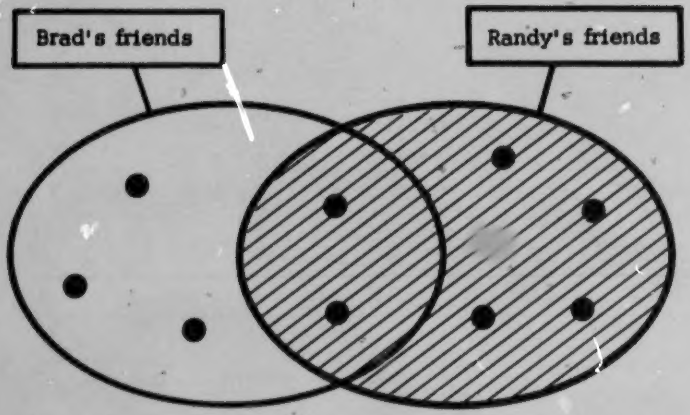
Brad has at least two friends.

Let students put some dots in the picture; two or more dots anywhere inside the blue string. The next statements you make should be adjusted to agree with what the students do. Suppose the class has fixed your picture as below.

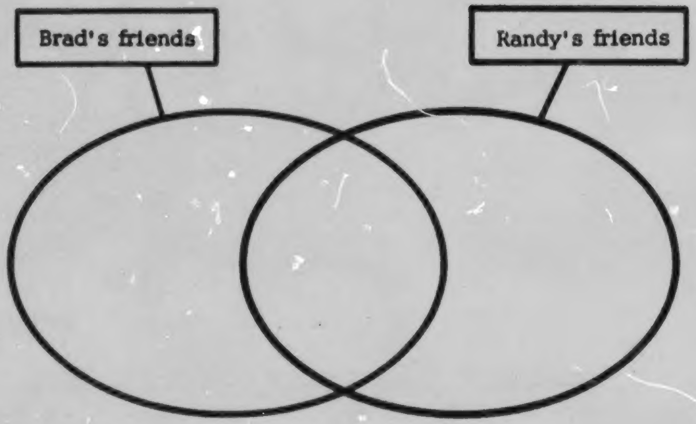

Brad's friends
Randy's friends

T: Randy has exactly six friends and at least two of them are also friends of Brad.

Let students experiment and discuss the problem. After some time you should have a picture like this one.


Brad's friends
Randy's friends

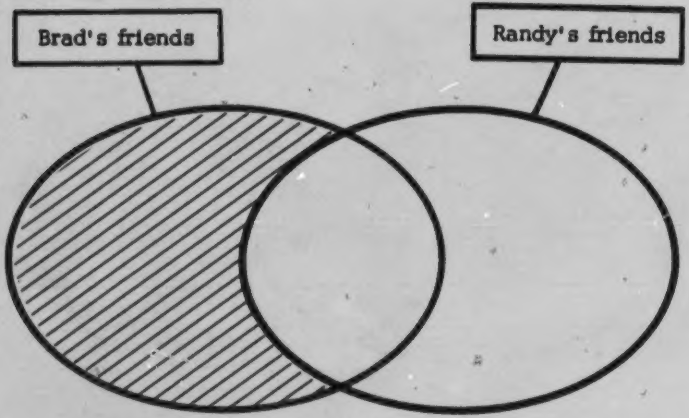
Return to the original picture.


Brad's friends
Randy's friends

T: Who can fix this picture so that this statement is true?

Every friend of Brad is also a friend of Randy.

Again, let students experiment and discuss each other's attempts. You may get several correct solutions; however, the one that gives the minimum information to make the above statement true is shown here.


Brad's friends
Randy's friends

T: Can we still fix this picture so that this statement is true?

Brad has at least two friends.

Students should add two or more dots to the middle region.

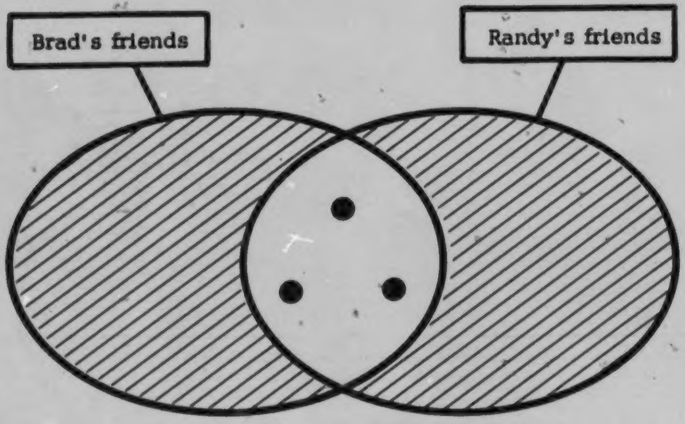
T: What other true statements can we make now?

S: Randy has at least two friends.

T: Fix the picture so that this statement is false.

Randy has a friend that is not a friend of Brad.

After some discussion your class should conclude this means that the statement "Every friend of Randy is also a friend of Brad" must be true. You should not insist that they verbalize this conclusion precisely, but rather they should discover that the picture should now have the right part inside the red string hatched.



Someone might comment that the picture now shows that Brad and Randy have the same friends. We could draw one string and label it with both tags.

If there is still interest in this situation, you could ask the class to give some other statements that are true; some other statements that are false; and some statements about which you can't tell.

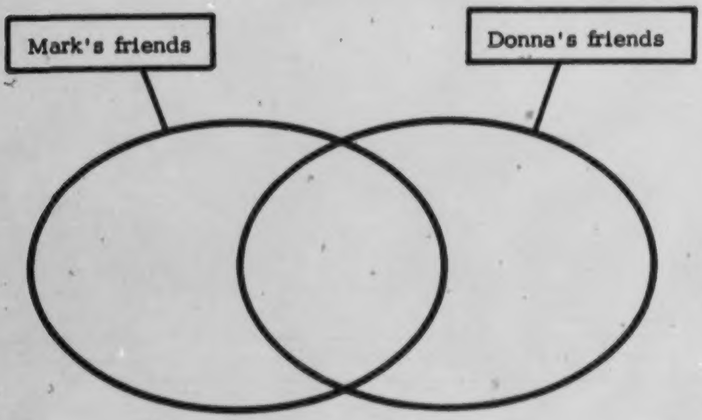
Exercise 3

Distribute paper and colored pencils.

Write these statements on the board.

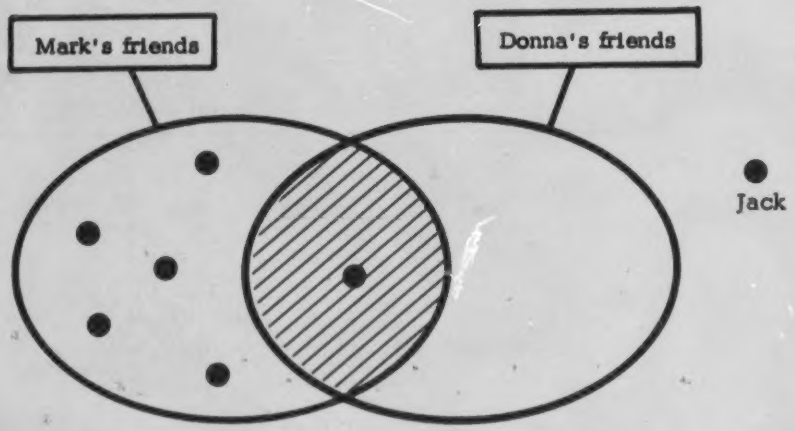
Mark has at least five friends.
Mark and Donna have exactly one friend in common.
Jack is not a friend of Mark and is not of friend of Donna.

Ask the students to draw a picture so that all of these statements are true. If the students need help getting started, draw the labeled strings on the board.



This will give you an opportunity to help students who are having difficulty with these situations. Students who finish quickly could be asked to write some of their own true statements, false statements, and "can't tell" statements about their pictures.

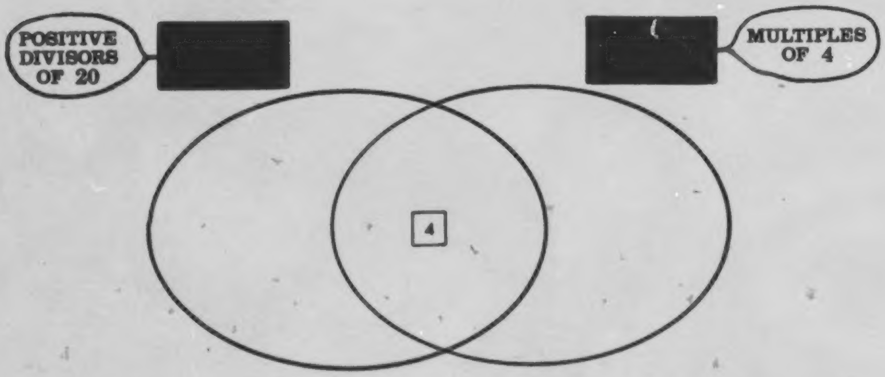
When most students have a picture, collectively draw a correct picture on the board.


Mark's friends
Donna's friends
Jack

2.4 String Game Analysis #2

Exercise 1

Distribute individual String Game Analysis Pad sheets to the students. Using two String Game demonstration analysis sheets from the Numerical String Game Kit, set up your board as below for a String Game. The bubbles indicate what is on the hidden tags.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: Today we are going to play the String Game. First, we will play altogether as a class and see how quickly we can discover what the strings are for. I have two starting clues which we will look at one at a time. The first clue is that 4 is in the center region. (Put " $\boxed{4}$ " on the board.) What information does this clue give us about the strings?

S: The red string cannot be for the multiples of 3, because 4 is not a multiple of 3.

S: Also, the blue string cannot be for the multiples of 3.

Cross out labels from the two lists as they are mentioned. Encourage your students to recognize that the same labels can be crossed out on both lists, because 4 is inside both strings; this clue gives the same information about both strings. After all the information from this clue has been discussed, the class should find that only seven possibilities remain on each list.

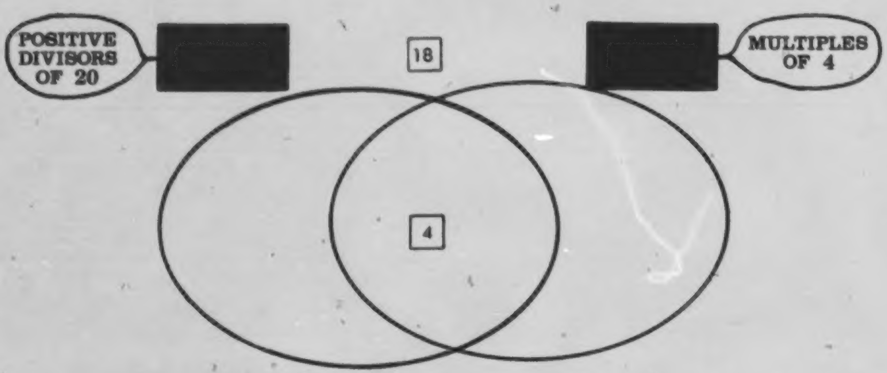
RED

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

BLUE

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
SMALLER THAN 50	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: The second clue is that 18 is in the outside region. (Put " $\boxed{18}$ " on the board.) What new information does this clue give us about the strings?



S: The red string cannot be for the multiples of 2, because 18 is a multiple of 2 and is outside the red string.

S: Also, the blue string cannot be for the multiples of 2.

Again, cross out labels from the two lists as they are mentioned. Encourage the students to again recognize that this clue gives the same information about both strings. Since 18 is outside of both strings, the same labels can be crossed out on both lists. The class should find that there are only four remaining possibilities for each of the strings.

RED

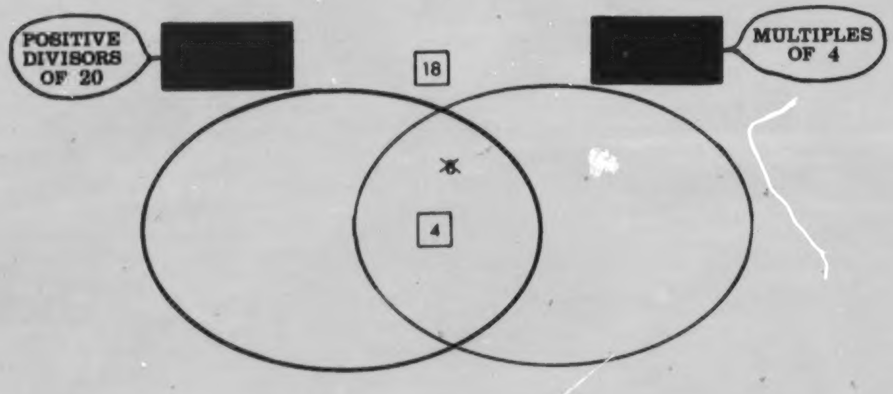
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: Now there are only four possibilities left for each string. Do you have just these four left on your lists, too? Let's play the game all together in the usual way and see how quickly we can discover what the strings are for. Who would like to make the first play?

Have students come and make plays, keeping count of how many plays are made. Accept any play, answering "yes" if the piece is played correctly and "no" if it is played incorrectly. Whenever a piece is played correctly, stop and analyze what new information was obtained. Make the appropriate crossing out of labels on the two lists. Whenever a piece is played incorrectly, record this information in the picture. For example, if " $\boxed{8}$ " is put in the center, indicate on your picture that it was put in the wrong region (see picture below).



CAUTION: Students might suggest that this play tells us that the red string cannot be for, say, the positive divisors of 24. Remind them that we still do not know where 8 belongs; it could still be inside the blue string, or inside the red string, or outside of both strings.

Continue playing the game altogether as a class until the strings are determined. You may wish to continue having students make plays even after the string labels are known just for the practice of recognizing positive divisors of 20 and multiples of 4.

Exercise 2

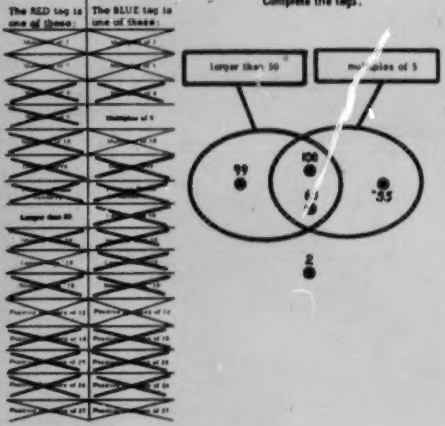
Play the Numerical String Game, Version D, in the usual way with the class divided into two teams. Encourage the students to use their lists (on the other side of the sheet from the individual String Game Analysis Pad) and eliminate possibilities for the string labels during the game. Suggest that when they are sure what the strings are for, they circle the correct label on each list. This will help them avoid mistakes when it is their turn.

Worksheets L17, L18, L19, and L20 are available for the rest of the lesson. If there is not time during this lesson for worksheets, you could instead use them at the end of the next String Game Analysis lesson, Section 2.5.

Name ____________________ L 17 *

Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you.

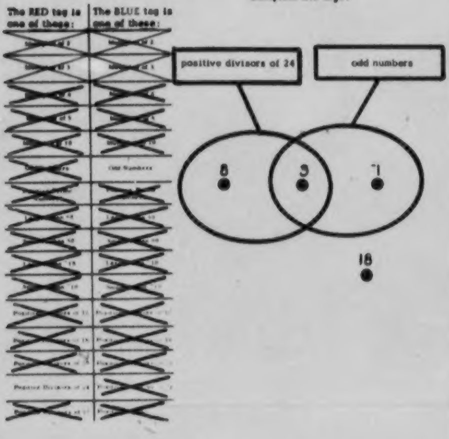
Complete the tags.


The RED tag is one of these:
The BLUE tag is one of these:
Larger than 50
Multiples of 5
Larger than 50
multiples of 5
99
100
-35
2

Name ____________________ L 18 **

Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you.

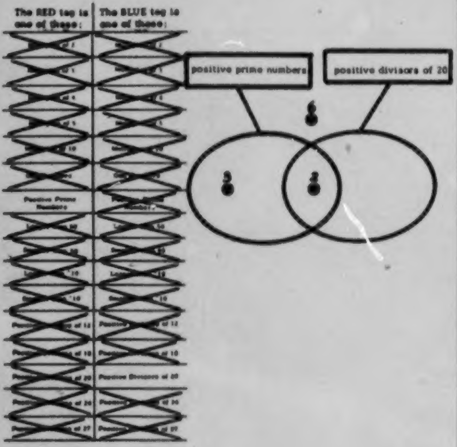
Complete the tags.


The RED tag is one of these:
The BLUE tag is one of these:
Odd Numbers
Positive Divisors of 24
positive divisors of 24
odd numbers
8
3
-1
18

Name ____________________ L 19 ***

Use the clues in the picture to cross out labels that the strings cannot have.

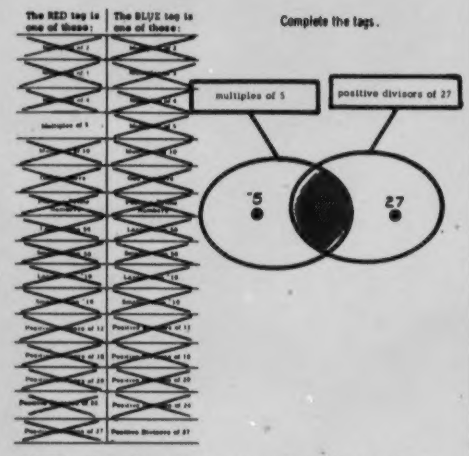
Complete the tags.


The RED tag is one of these:
The BLUE tag is one of these:
Positive Prime Numbers
Positive Divisors of 20
positive prime numbers
positive divisors of 20
6
3
2

Name ____________________ L 20 ****

Use the clues in the picture to cross out labels that the strings cannot have. The hatching is a clue.

Complete the tags.

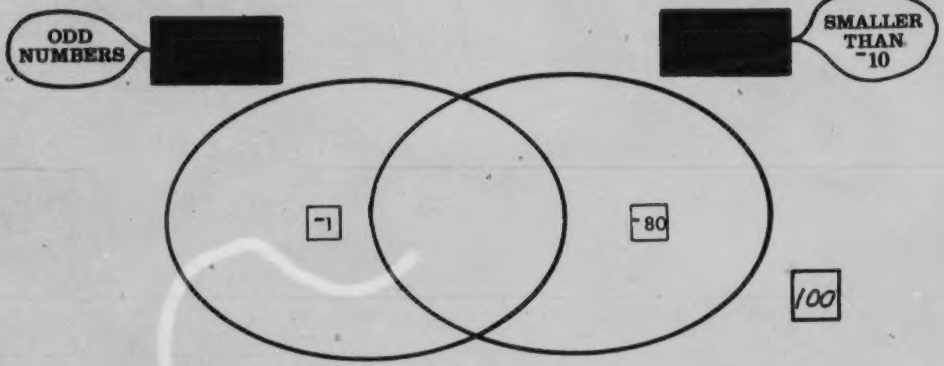

The RED tag is one of these:
The BLUE tag is one of these:
Multiples of 5
Positive Divisors of 27
multiples of 5
positive divisors of 27
-5
27

NOTE: Since the center region is empty (hatched), the blue string cannot be for the multiples of 3. If the blue string were for the multiples of 3, then 15 (for example) would be in the center region.

2.5 String Game Analysis #3

Exercise 1

Set up your board for the String Game and tape demonstration analysis sheets near the strings. Choose the starting clues so that in fact they determine the string labels. The situation pictured below is one such example. The bubbles show what is on the hidden tags.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN ⁻10
SMALLER THAN 50	SMALLER THAN ⁻10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN ⁻10
SMALLER THAN 50	SMALLER THAN ⁻10	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: What information do these clues give us about the strings?

Let the students suggest labels to cross out on the RED list and on the BLUE list. Have them give an explanation for each label that is crossed out. For example,

S: The red string cannot be for the numbers smaller than 50, because ⁻80 is smaller than 50 and is outside the red string.

S: The blue string cannot be for the positive divisors of any number, because $^{-}80$ is negative.

Cross out on the two lists labels that the strings cannot have as they are suggested by students. The students should find that all labels except one on each list can be crossed out. That is, these [redacted] clues determine the strings; the red string is for the odd numbers and the blue string is for the numbers smaller than $^{-}10$.

RED

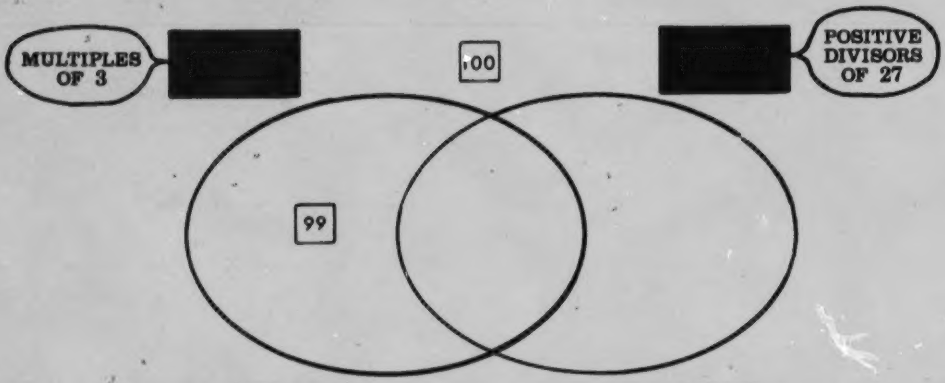
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	[✓ ODD NUMBERS]	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	[✓ SMALLER THAN $^{-}10$]	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

Exercise 2

Set up your board for a String Game with this starting situation.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: You are going to play the String Game all together as a class. The idea will be to see how many plays you need to make before you have determined the string labels.

We suggest that this exercise be treated as a kind of game. In our development classes, since there were two of them in the same school, we played one class against the other (the class using the fewest number of plays in order to determine the strings being the winner). If you teach two classes, or if there are two classes in your school at about the same place in the CSMP program, you could also play one class against the other. However, if you have just one class in the school at this place in the CSMP program, we suggest that you play your class against our classes (results from our classes are given below), or that you attempt to divide your class into two groups playing with half of the class while the other half is out of the room.

T: I have given you two clues to start. Let's see how much information they give us.

Let the students cross out as many labels as they can from the RED and BLUE lists. They should find that there are only two remaining possibilities for the red string, and there are eight remaining possibilities for the blue string.

RED

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	ODD NUMBERS	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
SMALLER THAN 50	SMALLER THAN $^{-}10$	~~ODD NUMBERS~~	POSITIVE PRIME NUMBERS

129

T: Now, you can make plays just as you do when we play the game. We will see how many plays you need to determine what the strings are for.

Keep a count of all the plays made; even those that get "no" answers. When a piece is played correctly, have the class analyze the new information. When a piece is played incorrectly, record this information in the picture. For example, if " $\boxed{^{-}1}$ " is played in the center region, the answer is "no". Remove the piece and write "$\cancel{^{-}1}$" in the center region.

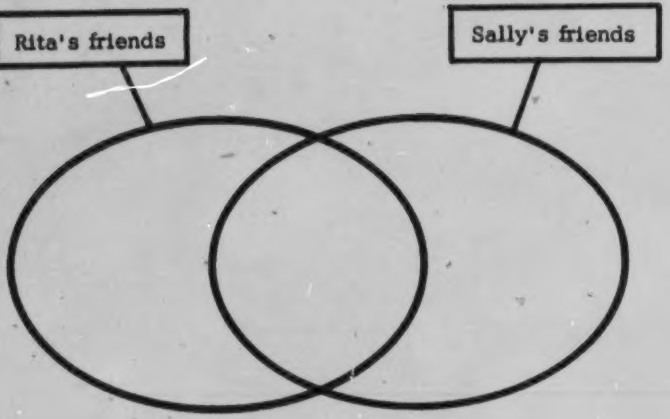
In our development classes using this same starting situation, one class used seven plays to determine the strings and the other used eight.

If you did not use Worksheets L17, L18, L19, and L20 with the lesson in Section 2.4, or if there was very little time to work on them then, you could have the students work on these worksheets for the rest of the lesson. Otherwise, if time permits, play the String Game, Version D and encourage the students to use the lists on a sheet from the individual String Game Analysis Pad.

2.6 Logical Thinking #2

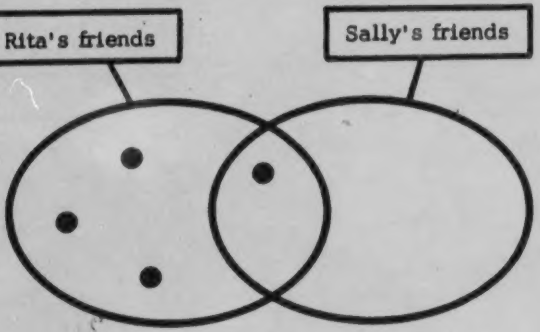
Exercise 1

Draw this string picture on the board.



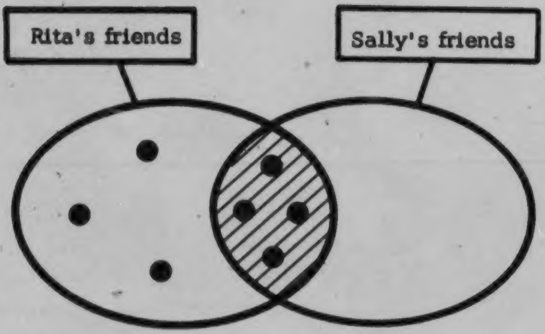
For each of the following statements, have students come to the board and fix the picture so that the statement is true. After each student response, let the class discuss whether or not it is correct and other possible ways to solve the problem. If necessary, reestablish the original picture before going on to the next statement. As suggested in the picture below it will not always be necessary to return to the original picture. Possible correct responses are shown at the right.

Rita has at least three friends.

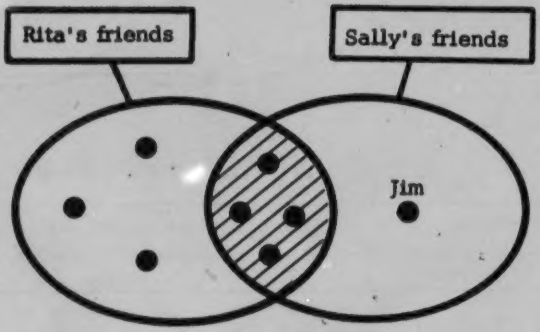


10L

Rita and Sally have exactly four friends in common.

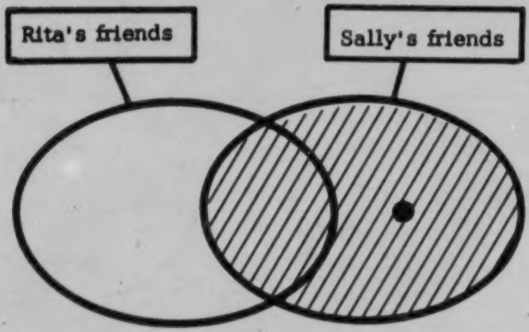

Rita's friends
Sally's friends

Jim is Sally's friend but not Rita's friend.

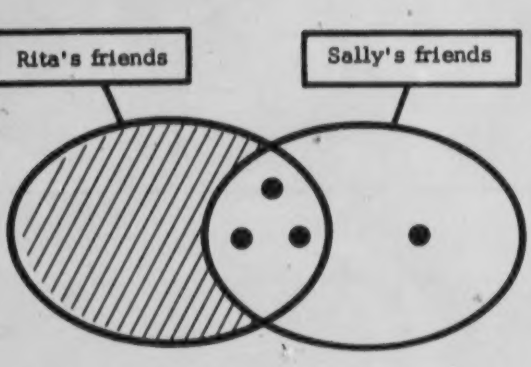

Rita's friends
Sally's friends
Jim

Reestablish the original picture before going on to the next statement.

Sally has at most two friends.

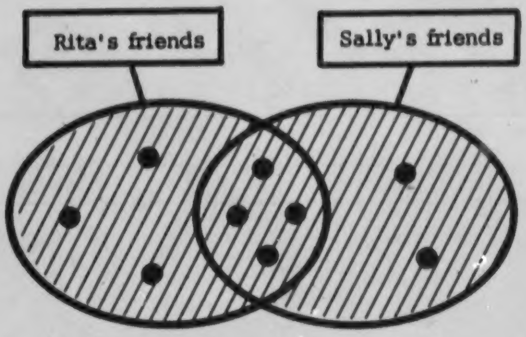

Rita's friends
Sally's friends

Every friend of Rita is a friend of Sally, but not every friend of Sally is a friend of Rita.



You could ask the class what is the fewest number of dots we need to put in the picture to make the last statement true. (Answer: One; inside the blue string and outside the red string)

Rita has exactly seven friends; Sally has exactly six friends; and altogether they have exactly nine friends.

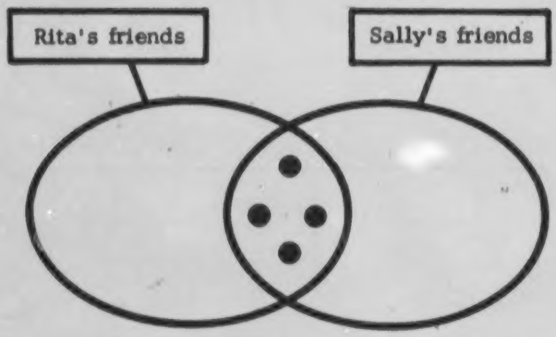


For this last statement you should allow the students to use trial and error methods. They probably will not immediately find the solution; however, by attempting to fix their own mistakes, they should soon arrive at a correct picture.

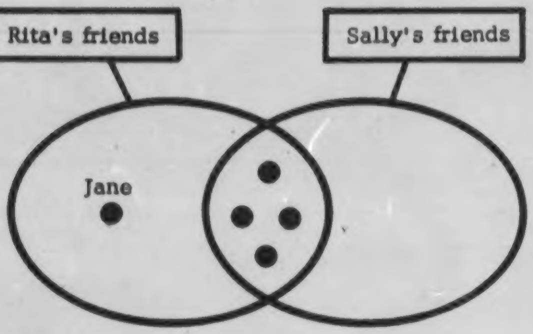
Return to the original picture. For each of the following statements, have students fix the picture so that the statement is false. Again, it may be necessary to reestablish the original picture before going on to the next statement. Possible correct responses are shown at the right.

10 L

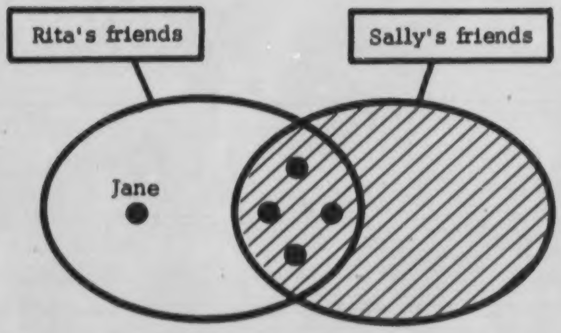
Rita and Sally have at most three friends in common.


Rita's friends
Sally's friends

Jane is Sally's friend.


Rita's friends
Sally's friends
Jane

Sally has at least five friends.


Rita's friends
Sally's friends
Jane

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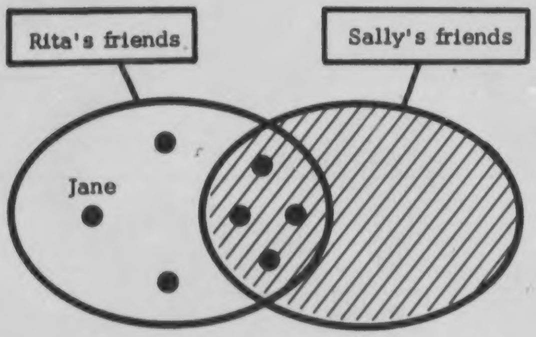
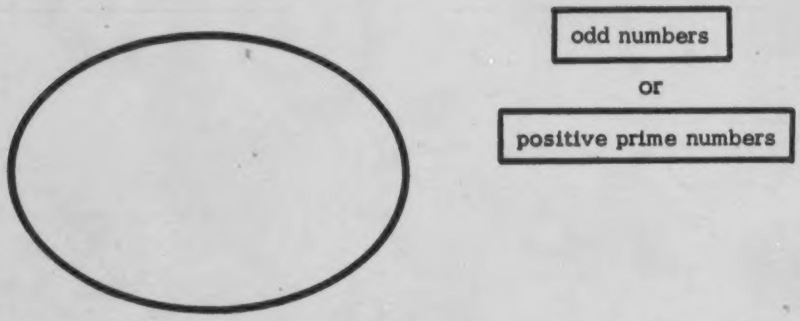
Rita has exactly one friend that is not a friend of Sally.


Rita's friends
Sally's friends
Jane

Exercise 2

Draw a red string on the board.

T: Imagine that we are playing the String Game and we have crossed out all the possibilities for the red string except: odd numbers, positive prime numbers. (Write these possibilities next to the string.)

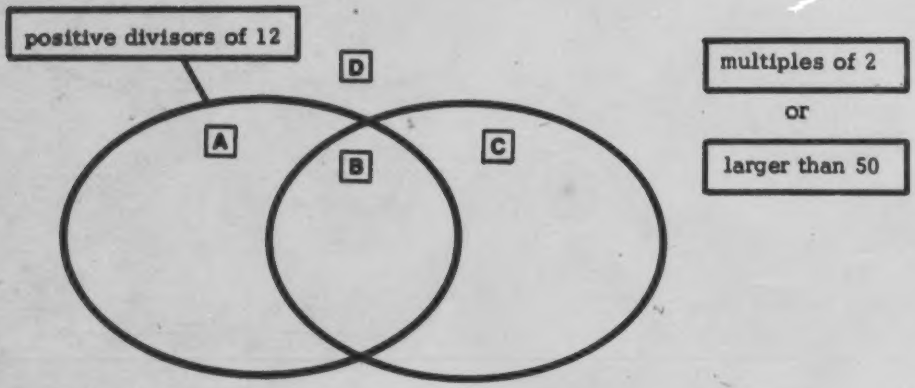

odd numbers
or
positive prime numbers

For each of the following numbers, have students decide whether it is inside the red string or outside the red string or whether you can't tell (CT).

4 (outside); 7 (inside); 27 (CT); 0 (outside); 2 (CT); ⁻2 (outside); ⁻55 (CT); 3 (inside); ⁻3 (CT); 105 (CT); 40 (outside).

Exercise 3

Draw this picture on the board.



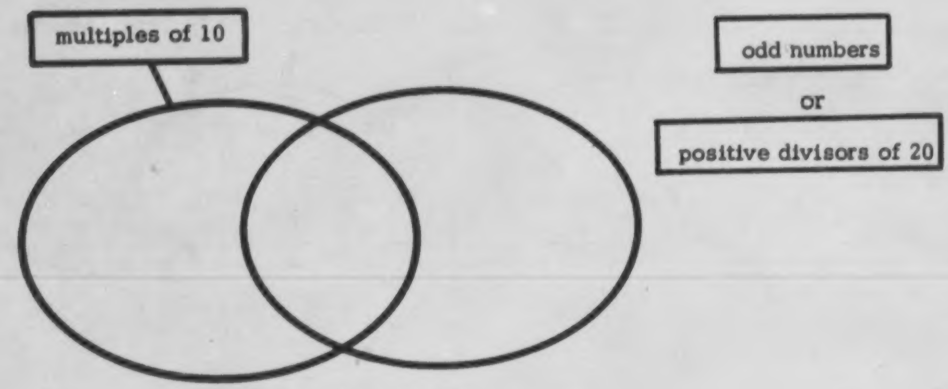
T: Again, imagine we are playing the String Game. This time we have found that the red string is for the positive divisors of 12, and we know that the blue string is for either the multiples of 2 or the numbers larger than 50 (i.e., suppose that all other possibilities have been eliminated).

For each of the following numbers, have students decide which region (A, B, C, or D) it is in or whether you can't tell (CT) which region it is in. For those that you can't tell, ask them to explain what are the possibilities.

9 (D); $^{-}80$ (CT; in C or D); 5 (D); 1 (A); 12 (CT; in A or B); 105 (CT; in C or D); 45 (D); $^{-}55$ (D); 3 (A); 0 (CT; in C or D).

Exercise 4

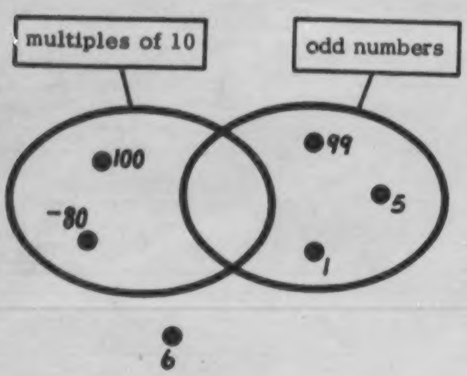
Draw this picture on the board.



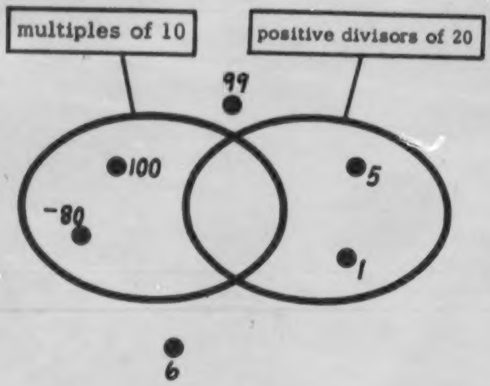
T: Again, imagine we are playing the String Game. We know that the red string is for the multiples of 10. We have eliminated all the possibilities for the blue string except: odd numbers, positive divisors of 20. I am going to give you six numbers. Five of these numbers you can put in the picture. One of them you can't tell where it is because we don't know what the blue string is for.

5; 6; 100; 1; 99; $^{-}80$

Let the students work independently on this problem for a couple minutes. Then begin to discuss it collectively. The given information is enough to know where to put 5, 6, 100, 1, and $^{-}80$. However, there are two choices for where to put 99; it could be inside the blue string and outside the red string, or it could be in the outside region. (See the next illustration.)


multiples of 10
odd numbers
100
-80
99
5
1
6

or


multiples of 10
positive divisors of 20
99
100
-80
5
1
6

T: What information do we get if 99 is inside the blue string and outside the red string?

S: Then the blue string is for the odd numbers, because 99 is an odd number but not a positive divisor of 20.

T: What information do we get if 99 is in the outside region?

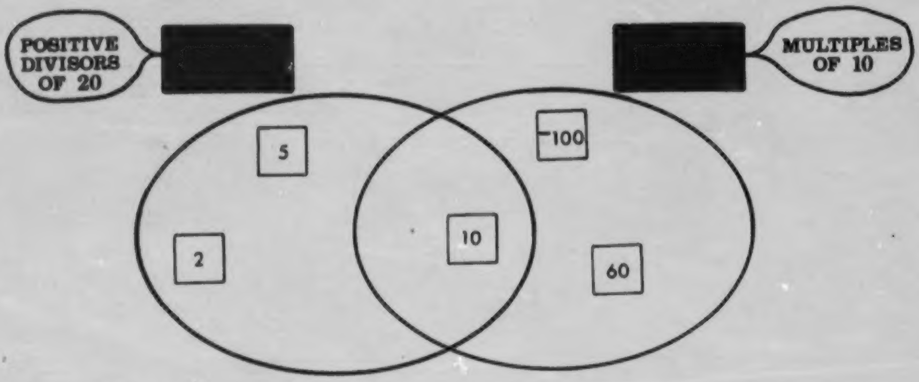
S: Then the blue string is for the positive divisors of 20.

If there is still time in your class period, you could play the String Game, Version D.

2.7 String Game Analysis #4

Exercise 1

Set up your board for the String Game and tape demonstration analysis sheets near the strings. Choose starting clues so that the strings are determined as illustrated below. The bubbles show what is on the hidden tags.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Have the students use the clues to cross out labels on the RED and BLUE lists. This activity should involve many different students and should go very quickly. You could sometimes let one student cross out several labels simultaneously. (For example, the blue string cannot have any of the "positive divisors of" labels because $^{-}100$ is negative and is inside the blue string.) The class should find that the clues determine the strings.

RED

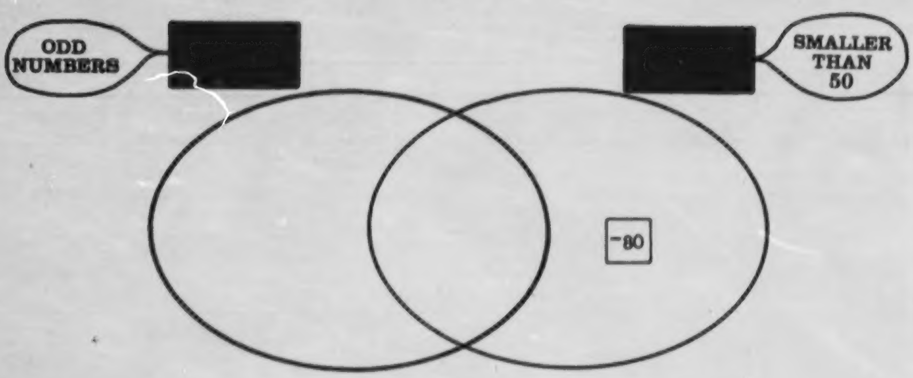
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	[✓ POSITIVE DIVISORS OF 20]
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}$10~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}$10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
[✓ MULTIPLES OF 10]	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}$10~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}$10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

Exercise 2

Play the game described in Section 2.5, Exercise 2 using the following starting situation.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}$10
SMALLER THAN 50	SMALLER THAN $^{-}$10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}$10
SMALLER THAN 50	SMALLER THAN $^{-}$10	ODD NUMBERS	POSITIVE PRIME NUMBERS

Before the class starts making plays, they should find they can cross out six labels on the RED list and ten labels on the BLUE list. Have them do this before you start counting the plays in the game.

RED

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
SMALLER THAN 50	SMALLER THAN $^{-}10$	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

In this same situation, both of our development classes used eight plays in order to determine the strings.

Exercise 3

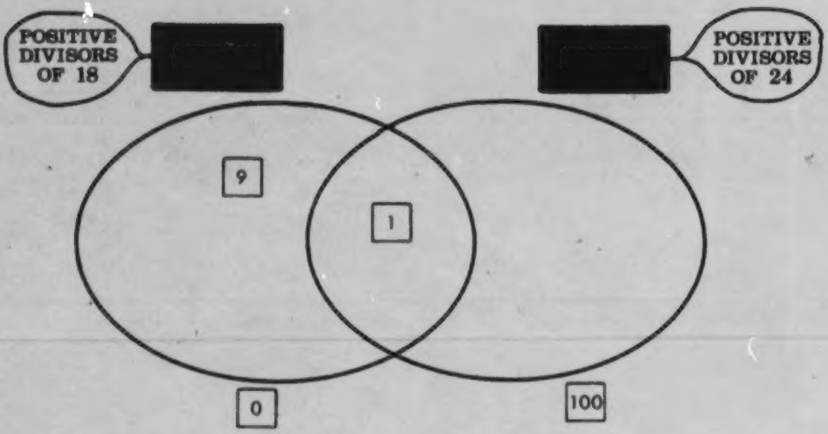
Distribute sheets from the individual String Game Analysis Pad.

T: Today we are going to play the String Game with some new scoring rules. First we divide into two teams, A and B (do this). Then, you write your name and your team (A or B) at the top of your lists.

Explain the new scoring rules as described in the appendix to this chapter, Version E: The String Game with Special Scoring Rules.

The first time the String Game is played with these special scoring rules, give several starting clues so as to increase the chance of some students having determined the strings after one or two rounds. This will make the game more exciting. We suggest you could begin with this situation having four starting clues.

11L



In this case, before play begins, the students could cross out all except three labels on each of the RED and BLUE lists. You will not do this at the board. Students should do this on their own; it is part of the game.

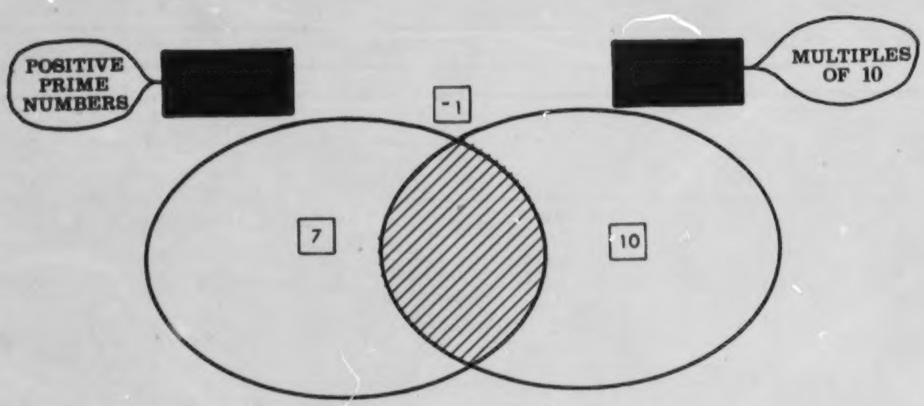
RED	BLUE
~~Multiples of 2~~	~~Multiples of 2~~
~~Multiples of 3~~	~~Multiples of 3~~
~~Multiples of 4~~	~~Multiples of 4~~
~~Multiples of 5~~	~~Multiples of 5~~
~~Multiples of 10~~	~~Multiples of 10~~
Odd Numbers	~~Odd Numbers~~
~~Positive Prime Numbers~~	~~Positive Prime Numbers~~
~~Larger than 50~~	~~Larger than 50~~
~~Smaller than 50~~	~~Smaller than 50~~
~~Larger than $^-10$~~	~~Larger than $^-10$~~
~~Smaller than $^-10$~~	~~Smaller than $^-10$~~
~~Positive Divisors of 12~~	Positive Divisors of 12
Positive Divisors of 18	~~Positive Divisors of 18~~
~~Positive Divisors of 20~~	Positive Divisors of 20
~~Positive Divisors of 24~~	Positive Divisors of 24
Positive Divisors of 27	~~Positive Divisors of 27~~

2.8 String Game Analysis #5

Distribute sheets from the individual String Game Analysis Pad to the students. Since these sheets are printed on both sides with lists of the possible string labels, one sheet should suffice for both exercises in this lesson.

Exercise 1

Set up your board for a String Game and tape demonstration analysis sheets near the strings. Use the starting situation illustrated below. In this case, one clue is that the center region is empty, and this clue together with the three number clues determine the strings. The bubbles show what is on the hidden tags.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: Let's look first at the number clues. What information do they give us about the strings?

Have the students work independently for a few minutes on this problem, crossing out on their lists as many labels as they can. Then, collectively let students suggest labels to cross out on the RED and BLUE lists. Be sure adequate explanations are given for crossing out labels. They should find that the three number clues determine the red string and leave four possibilities for the blue string.

RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: Your lists should look like these. Circle "Positive Prime Numbers" on the RED list, because we now know the red string is for the positive prime numbers. There are still four possibilities for the blue string. But we have another clue. The center region is empty. What information does this clue give us about the blue string?

S: The blue string cannot be for the multiples of 2, because if it were, 2, a prime number, would be in the center region.

This clear an explanation will not be easy for students to make at first. Let them struggle for a short while with the situation. If no one suggests some label to cross out on the BLUE list, ask the class to consider the situation with the red string for the positive prime numbers and the blue string for the multiples of 2 (or 5).

S: The blue string cannot be for the multiples of 5, because if it were, 5, a prime number, would be in the center region.

S: The blue string cannot be for the positive divisors of 20, because if it were, 2 (or 5), a prime number, would be in the center region.

Cross out these labels on the BLUE list. The students should conclude that the blue string is for the multiples of 10.

RED

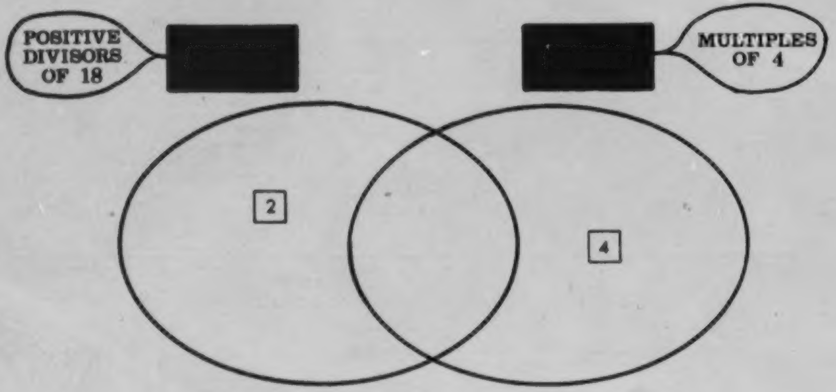
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	[✓ POSITIVE PRIME NUMBERS]

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
[✓ MULTIPLES OF 10]	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

Exercise 2

Set up your board for a String Game and tape demonstration analysis sheets near the strings. Start with two clues as shown below.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN ⁻10
SMALLER THAN 50	SMALLER THAN ⁻10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN ⁻10
SMALLER THAN 50	SMALLER THAN ⁻10	ODD NUMBERS	POSITIVE PRIME NUMBERS

Have the students independently cross out on their RED and BLUE lists as many labels as they can. After a few minutes, do the analysis with these clues collectively. The class should find that there are two remaining possibilities for the red string and that the blue string is determined.

RED

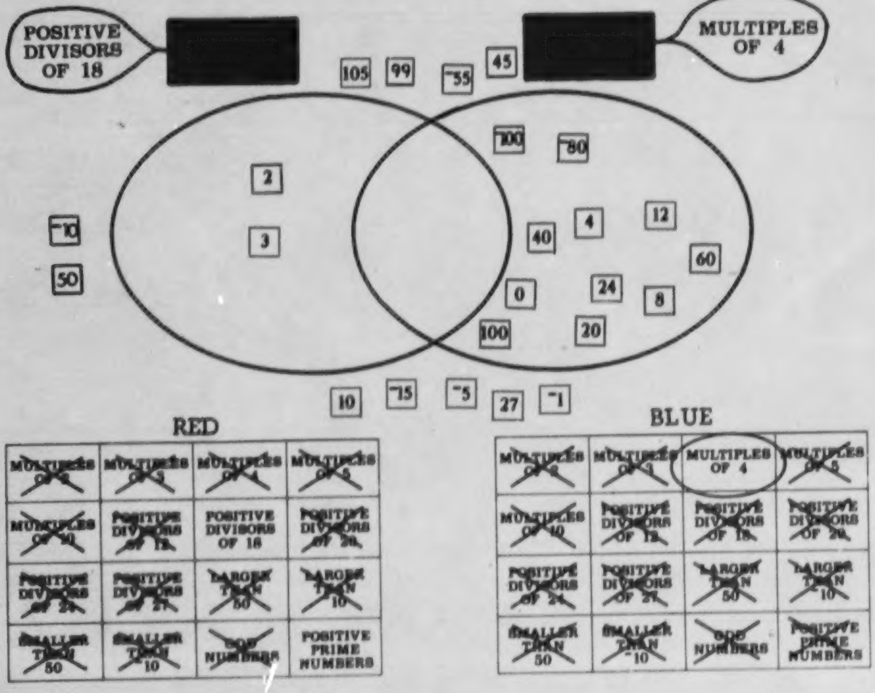
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN ⁻10~~
~~SMALLER THAN 50~~	~~SMALLER THAN ⁻10~~	~~ODD NUMBERS~~	POSITIVE PRIME NUMBERS

BLUE

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	[✓ MULTIPLES OF 4]	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN ⁻10~~
~~SMALLER THAN 50~~	~~SMALLER THAN ⁻10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: Now we know that the red string is for either the positive divisors of 18 or for the positive prime numbers. We also know that the blue string is for the multiples of 4. Let's see how many of the numbers from the String Game we can put in the picture even though we are not yet sure what the red string is for.

Let students put as many of the String Game numbers as they can in the picture. This activity should allow everyone a chance to participate. Encourage discussion about why, for example, they are sure that 10 is in the outside region (Answer: 10 is not a positive divisor of 18 and not a positive prime number and not a multiple of 4); or why they are sure that 3 is inside the red string but outside the blue string (Answer: 3 is a positive divisor of 18 and a positive prime number, but 3 is not a multiple of 4); and so on. The students should put all the String Game numbers except 1, 5, 6, 7, 9, and 18 in the picture.


POSITIVE DIVISORS OF 18
MULTIPLES OF 4
105
99
⁻35
45
⁻100
⁻80
2
3
⁻10
50
40
4
12
60
0
24
8
100
20
10
⁻15
⁻5
27
⁻1
RED
MULTIPLES OF 2
MULTIPLES OF 3
MULTIPLES OF 4
MULTIPLES OF 5
MULTIPLES OF 10
POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27
LARGER THAN 50
LARGER THAN 10
SMALLER THAN 50
SMALLER THAN 10
ODD NUMBERS
POSITIVE PRIME NUMBERS
BLUE
MULTIPLES OF 2
MULTIPLES OF 3
MULTIPLES OF 4
MULTIPLES OF 5
MULTIPLES OF 10
POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27
LARGER THAN 50
LARGER THAN ⁻10
SMALLER THAN 50
SMALLER THAN ⁻10
ODD NUMBERS
POSITIVE PRIME NUMBERS

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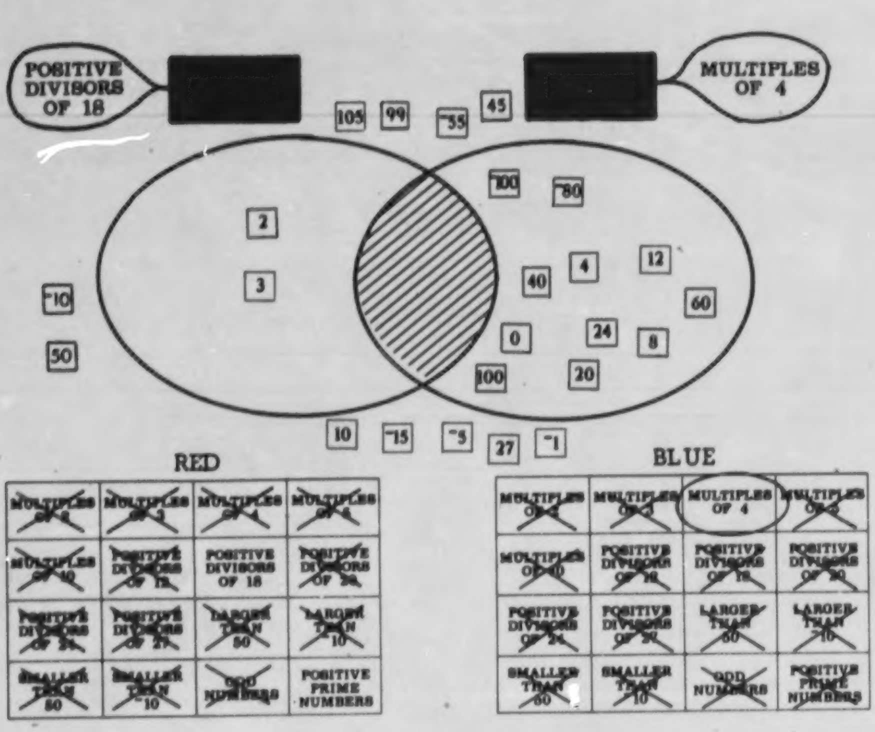
Discuss the possibilities for the numbers 1, 5, 6, 7, 9, and 18. Each of these numbers could be in the outside region, or inside the red string and outside the blue string.

T: Do you see a region that should be hatched because it is empty?

S: The center region.


POSITIVE DIVISORS OF 18
MULTIPLES OF 4
105
99
⁻35
45
2
3
⁻100
⁻80
4
12
40
60
0
24
8
100
20
⁻10
50
10
⁻15
⁻5
27
⁻1
RED
BLUE
MULTIPLES OF 2
MULTIPLES OF 3
MULTIPLES OF 4
MULTIPLES OF 5
MULTIPLES OF 10
POSITIVE DIVISORS OF 18
POSITIVE DIVISORS
LARGER THAN 50
LARGER THAN ⁻10
SMALLER THAN 50
SMALLER THAN ⁻10
ODD NUMBERS
POSITIVE PRIME NUMBERS

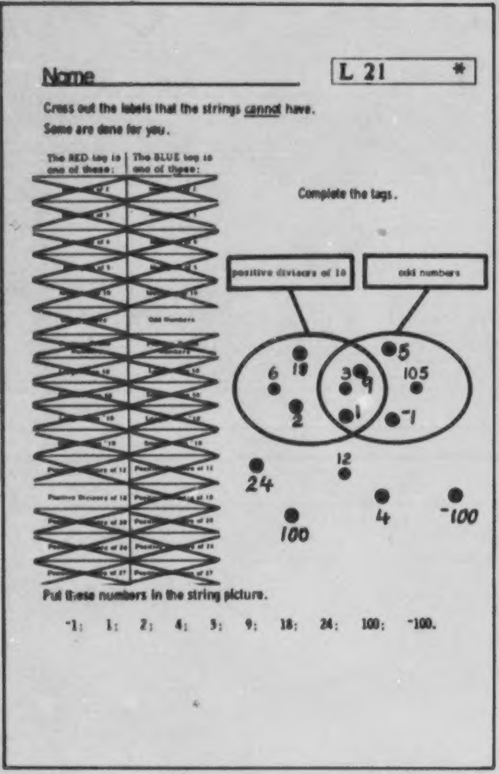
Take down all the game pieces except " $\boxed{2}$ " and " $\boxed{4}$ " ; that is, return to the original situation.

T: Suppose it is your turn to play in the String Game, and you would like to determine the red string as quickly as possible. Which piece would you play?

Give the students a few minutes to think about which play would be helpful. They should, after some trial and error and class discussion, find that playing one of the numbers 1, 5, 6, 7, 9, or 18 would give them the necessary information (as long as they did not make a careless play). For example, if "$\boxed{5}$" were played in the outside region and given a "no" answer, then you know it belongs inside the red string so that the red string would be for the positive prime numbers. If "$\boxed{5}$" played in the outside region got a "yes" answer, then you know that the red string is for the positive divisors of 18. A similar analysis applies to each of the numbers 1, 6, 7, 9, and 18.

Let the students make one of these plays to determine the red string.

Worksheets L21 and L22 are available for the rest of the lesson. If you have time remaining, play the Numerical String Game with the special scoring rules, Version E.



Name

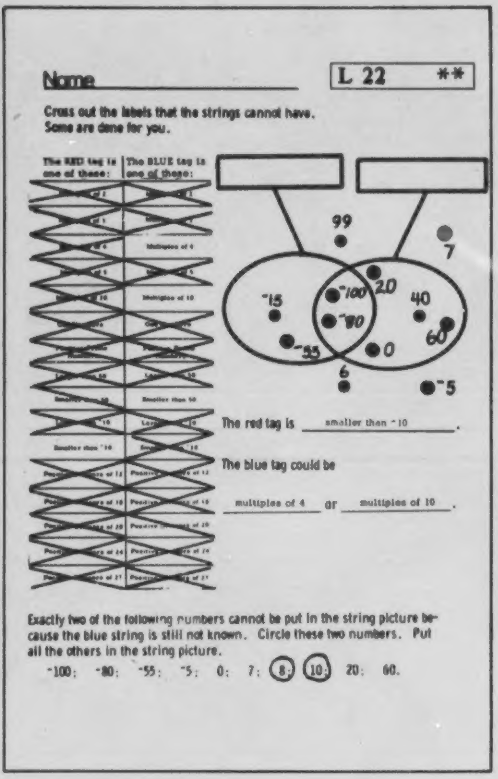
L 21 *

Cross out the labels that the strings cannot have.
Some are done for you.

The RED tag is one of these:
The BLUE tag is one of these:
Odd Numbers
Positive Divisors of 18
Complete the tags.
positive divisors of 18
odd numbers
6
18
2
3
9
1
5
105
⁻1
12
24
100
4
⁻100

Put these numbers in the string picture.

⁻1; 1; 2; 4; 5; 9; 18; 24; 100; ⁻100.



Name

L 22 **

Cross out the labels that the strings cannot have.
Some are done for you.

The RED tag is one of these:
The BLUE tag is one of these:
Multiples of 4
Multiples of 10
Smaller than 10
Smaller than 10
Smaller than ⁻10
99
7
⁻15
⁻100
⁻80
⁻55
20
40
60
0
6
⁻5

The red tag is smaller than ⁻10.

The blue tag could be

multiples of 4 or multiples of 10.

Exactly two of the following numbers cannot be put in the string picture because the blue string is still not known. Circle these two numbers. Put all the others in the string picture.

⁻100; ⁻80; ⁻55; ⁻5; 0; 7; 8; 10; 20; 60.

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APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of numbers make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks or using numbers, and we begin the process of detailing the various possibilities open to you when you play the game. The different variations of the game described in this appendix are referred to in several of the lessons in this chapter. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that it is most beneficial (and enjoyable) for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

PLAYING BOARD

The most easily manageable equipment for this game depends upon your having a magnetic* chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

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* Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic".

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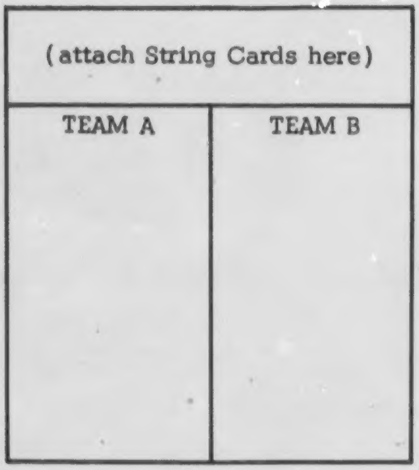
GAME BOARD

The Game Board is divided into three regions all illustrated below.



a) Magnetic: If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Game Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Game Board could be drawn on the board. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size: 60 cm by 80 cm) or locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Game Board. In this case draw the markings for the three regions of the Game Board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

b) Non-magnetic: If you do not have a magnetic chalkboard available for the playing board, then your Game Board could be a large piece of poster board (minimum size: 60 cm by 80 cm). Make the markings for the three regions of the Game Board with a heavy felt-tip pen.

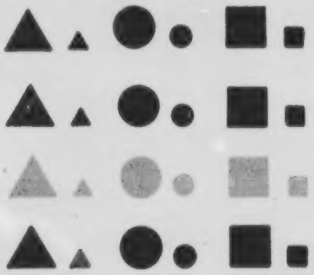
GAME PIECES and STRING CARDS

One set of game pieces and two sets of string cards are needed for each version of the game. One set of the string cards is the "face-up" set and is to be attached to the upper section of the Game Board - it is a constant reminder during the game of the possible labels for the strings. The other set is the "face-down" set.

Game Pieces

String Cards

String Game with A-Blocks



△	○	□	RED	BLUE	GREEN	YELLOW	BIG
NOT △	NOT ○	NOT □	NOT RED	NOT BLUE	NOT GREEN	NOT YELLOW	LITTLE

String Game with Numbers: Multiples and Order

⁻100	⁻80	⁻55	⁻20	⁻15	⁻12
⁻6	⁻5	⁻2	⁻1	0	1
7	10	15	24	27	30
40	45	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5	MULTIPLES OF 10
ODD NUMBERS	LARGER THAN 50	SMALLER THAN 50	LARGER THAN ⁻10	SMALLER THAN ⁻10

String Game with Numbers: Multiples, Divisors, and Order

⁻100	⁻80	⁻55	⁻15	⁻10	⁻5
⁻1	0	1	2	3	4
5	6	7	8	9	10
12	18	20	24	27	40
45	50	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN ⁻10
SMALLER THAN 50	SMALLER THAN ⁻10	ODD NUMBERS	POSITIVE PRIME NUMBERS

Game pieces and string cards for the String Game with Numbers, all versions, can be found in the Numerical String Game Kit. You will have to note which number cards and string cards to use with the particular version you are playing.

a) Magnetic: The game pieces (A-Blocks or number cards) can be magnetized by sticking to the back of each one a small piece of magnetic material. (Magnetic material is included in the Numerical String Game Kit, or it is available in many hobby and notions departments of stores.) To each string card in the "face-down" set, stick a small piece of magnetic material to the front of the card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces and string cards can be attached to the Game Board by means of loops of masking tape stuck to the backs. A "face-down" string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card from the face-down set. Stand the Game Board on the chalk tray, or have it near and convenient to the playing board. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces on the string picture. This removes from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place on the game board before the game begins.

Object of the Game

Each team tries to place all of its game pieces correctly (according to the face-down string cards) on the string picture. The winning team is the one which first places all of its game pieces correctly and identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game

1) The game is played in silence. This is to avoid one student telling another where to play a certain game piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes". The piece remains on the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no". The player returns the piece to the team's unplayed collection and play passes to the other team.

   As an aid in judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace on the string picture a correctly placed piece that had been rejected because you had disapproved its position.

4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards.

If these are all correct, the team has won. If a mistake is made (even if it is only in the case of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces, and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

Version A: The String Game with A-Blocks

This simplest version of the game uses 24 A-Blocks as game pieces and only nine string cards.

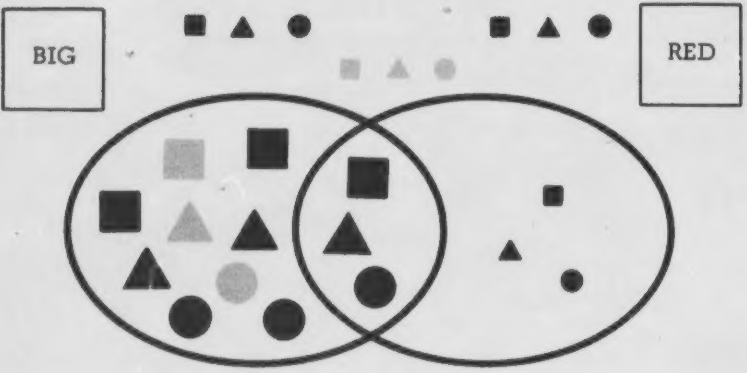
△	○	□	RED	BLUE	YELLOW	GREEN	BIG	LITTLE

(See the description under "Equipment" of game pieces and the two sets of string cards needed for the game.)

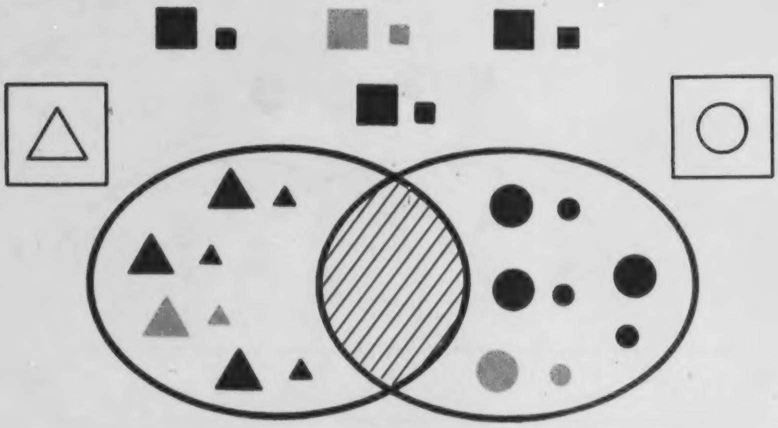
We provide here several example "crib-sheets" for variations of the game with two and three strings.

Variation #1: Two Strings

Example 1: No Empty Regions

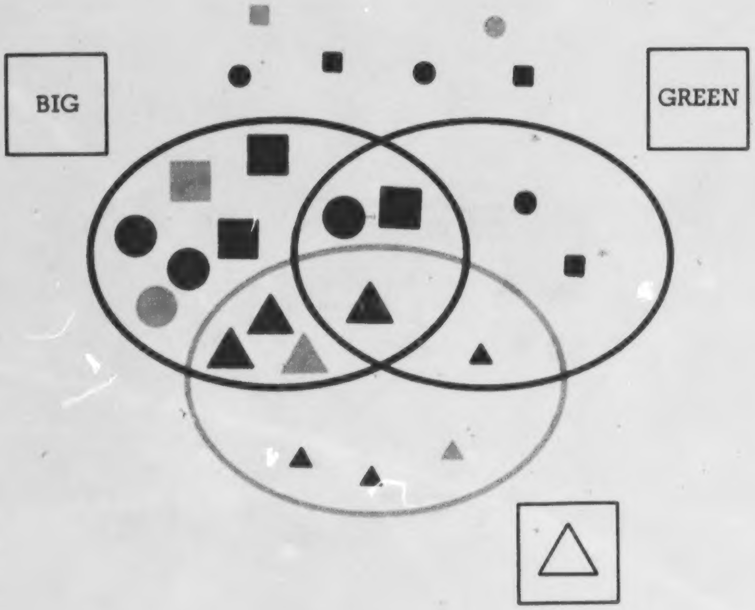


Example 2: One Empty Region

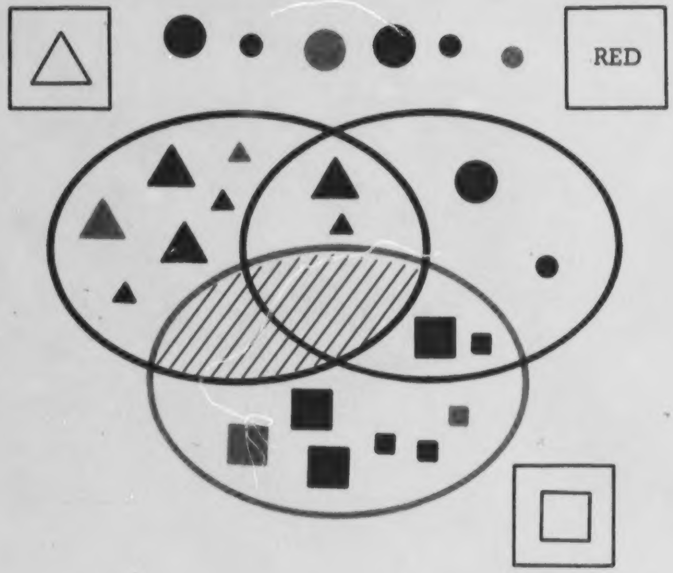


Variation #2; Three Strings

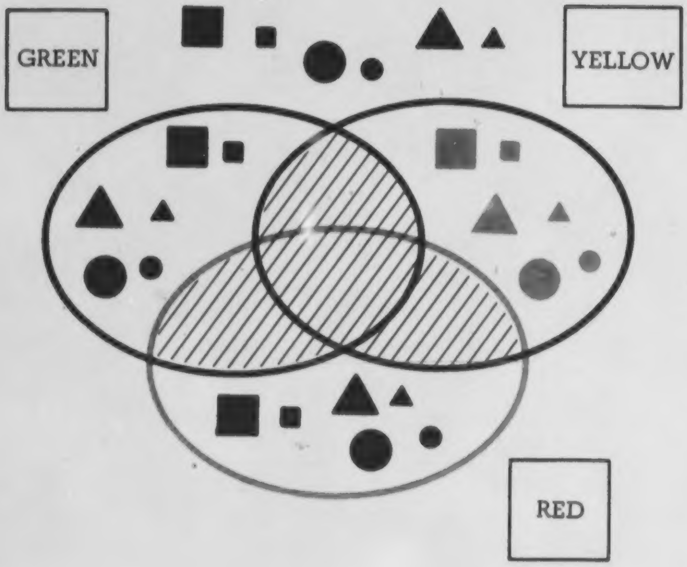
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions


GREEN
YELLOW
RED

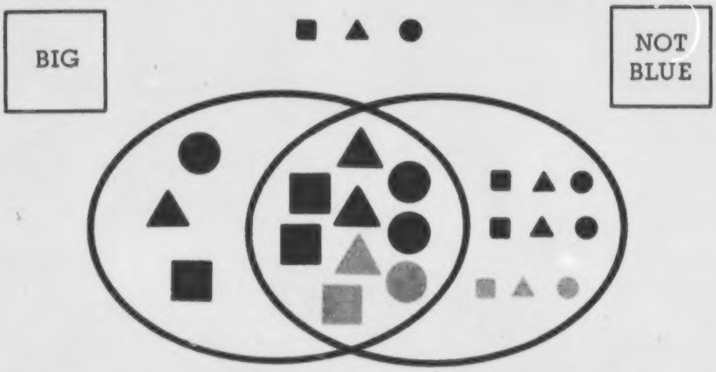
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all 16 of the string cards. (See the description under "Equipment" of game pieces and the two sets of string cards needed for the game with A-Blocks.)

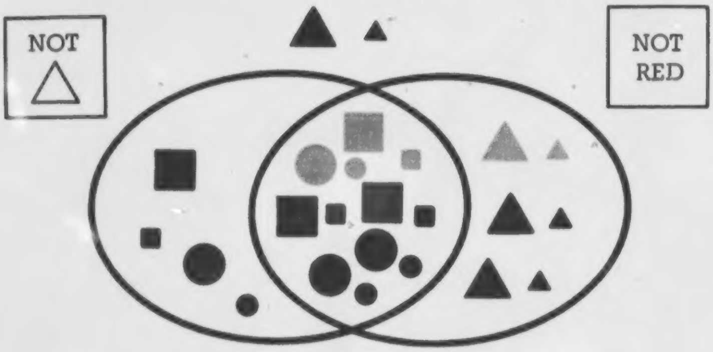
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss that it is unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

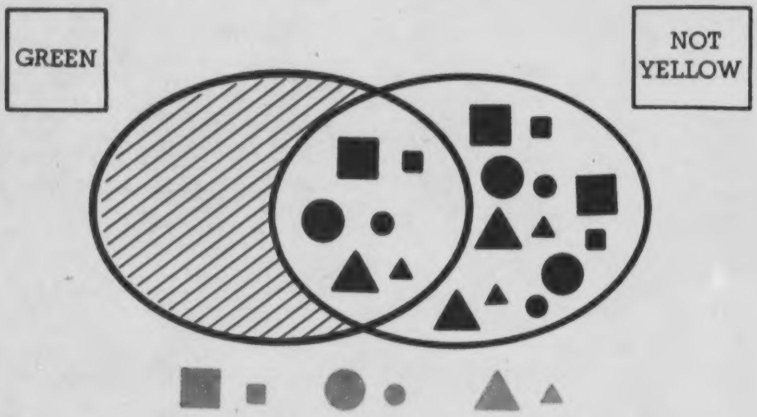
Example 1: No Empty Regions



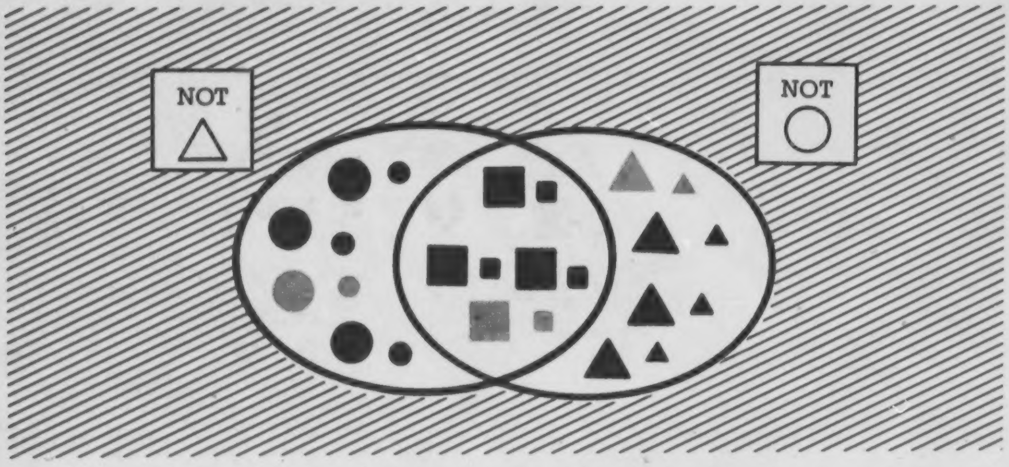
Example 2: No Empty Regions



Example 3: One Empty Region



Example 4: One Empty Region



Of course, by hatching the "outside" region of this diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

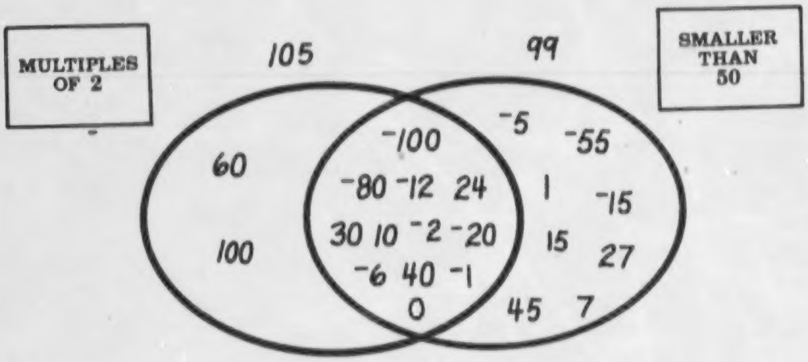
Version C: The String Game with Numbers; Multiples and Order

This version of the game uses 24 number cards as game pieces and two sets of ten string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string.

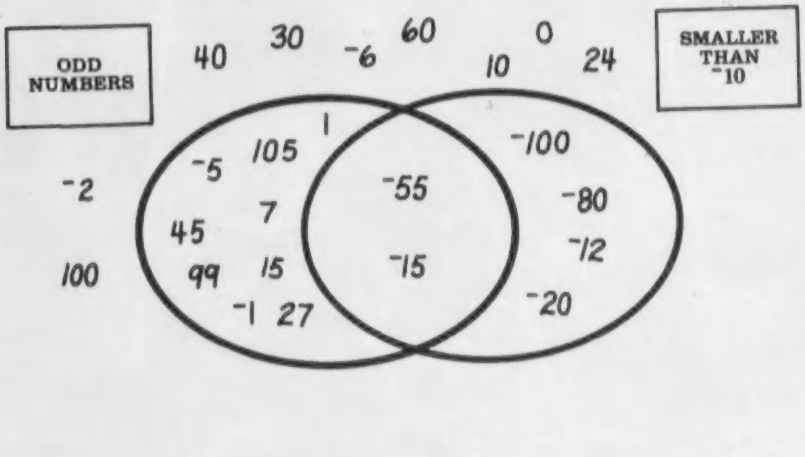
We provide here several example "crib-sheets" for variations of this game using two and three strings.

Variation #1: Two Strings

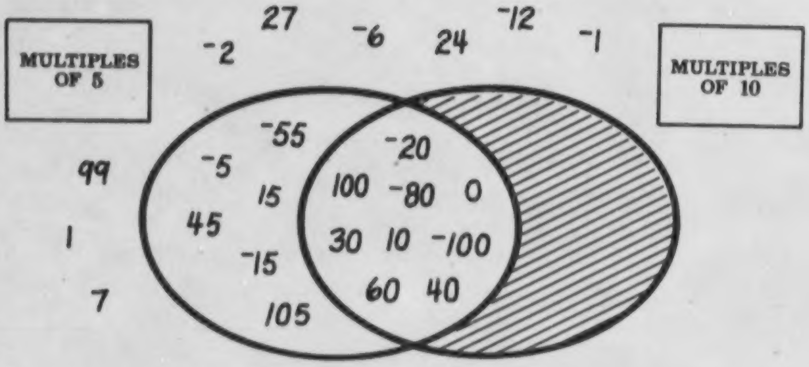
Example 1: No Empty Regions


MULTIPLES OF 2
105
99
SMALLER THAN 50
60
100
-100
-80 -12 24
30 10 -2 -20
-6 40 -1
0
-5
-55
1
-15
15
27
45
7

Example 2: No Empty Regions


ODD NUMBERS
40
30
-6
60
0
10
24
SMALLER THAN -10
-2
100
1
-5
105
7
45
99
15
-1
27
-55
-15
-100
-80
-12
-20

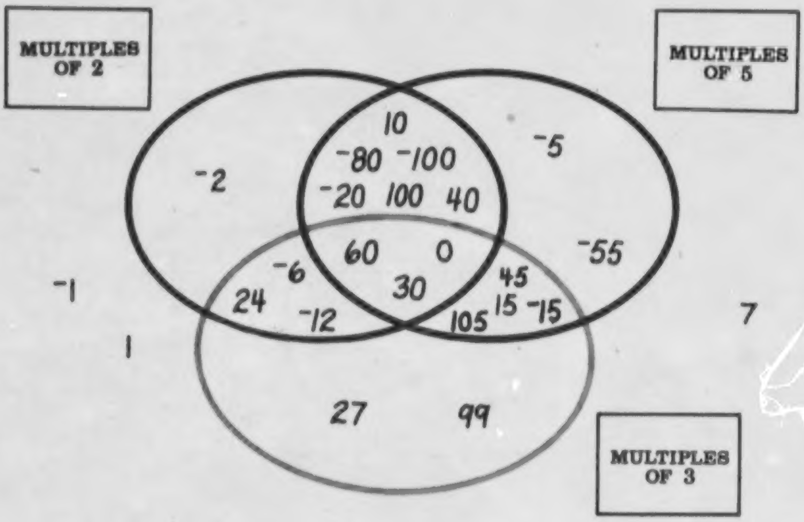
Example 3: One Empty Region


MULTIPLES OF 5
MULTIPLES OF 10
27
-2
-6
24
-12
-1
99
1
7
-55
-5
15
45
-15
105
-20
100
-80
0
30
10
-100
60
40

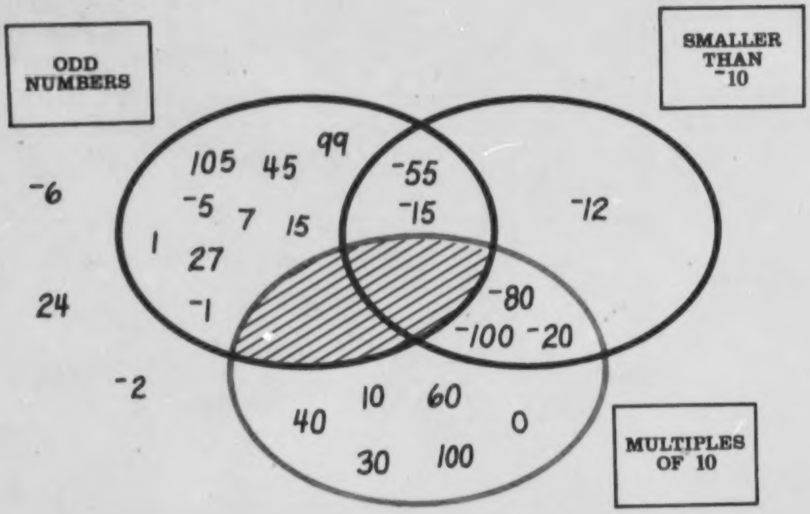
Variation #2: Three Strings

CAUTION: The jump in difficulty from the two-string variation to the three-string variation is much greater in the case of Version C of the String Game than it was in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

Example 1: No Empty Regions


MULTIPLES OF 2
MULTIPLES OF 5
MULTIPLES OF 3
10
-80
-100
-20
100
40
-2
-5
-55
60
0
30
-6
24
-12
45
15
-15
105
-1
1
7
27
99

Example 2: Two Empty Regions

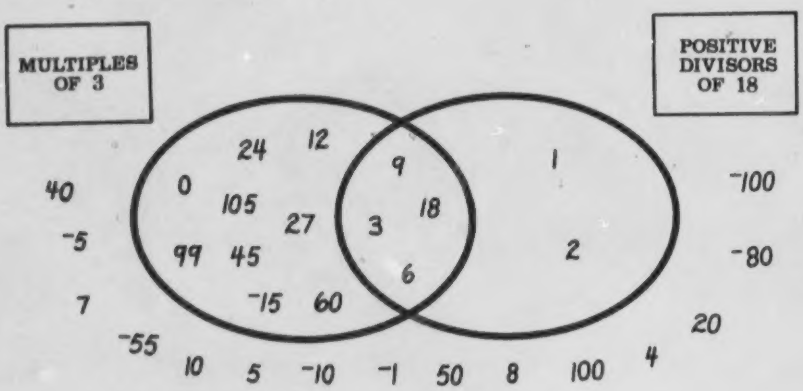

ODD NUMBERS
SMALLER THAN -10
MULTIPLES OF 10
-6
24
-2
105
45
99
-5
7
15
1
27
-1
-55
-15
-12
-80
-100
-20
10
60
40
0
30
100

Version D: The String Game with Numbers: Multiples, Divisors, and Order

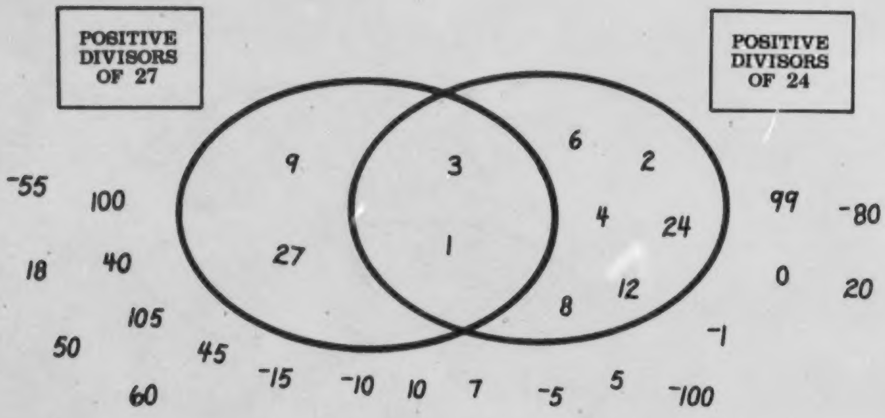
This version of the game uses 30 number cards as game pieces and two sets of 16 string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string. Two examples of this are provided in "crib-sheet" examples provided here; however, before using such a game the players should have previously understood that this was a possibility.

Variation #1: Two Strings

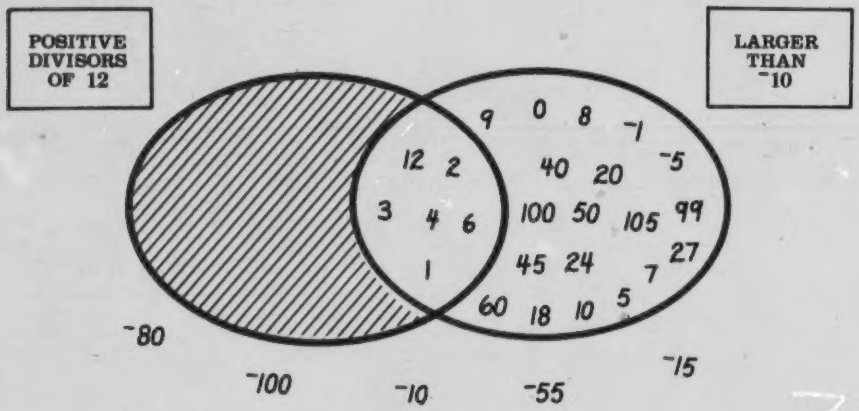
Example 1: No Empty Regions


MULTIPLES OF 3
POSITIVE DIVISORS OF 18
40
-5
7
-55
24
12
0
105
27
99
45
-15
60
9
18
3
6
1
2
-100
-80
20
4
10
5
-10
-1
50
8
100

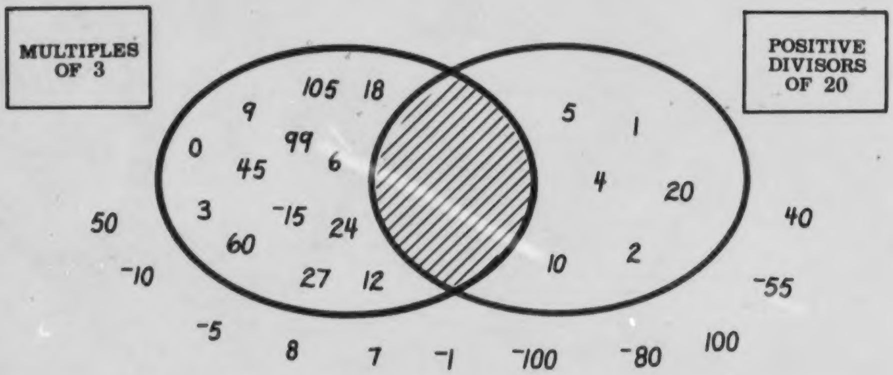
Example 2: No Empty Regions


POSITIVE DIVISORS OF 27
POSITIVE DIVISORS OF 24
9
27
3
1
6
2
4
24
12
8
⁻55
100
18
40
105
50
45
60
⁻15
⁻10
10
7
⁻5
5
⁻100
⁻1
99
⁻80
0
20

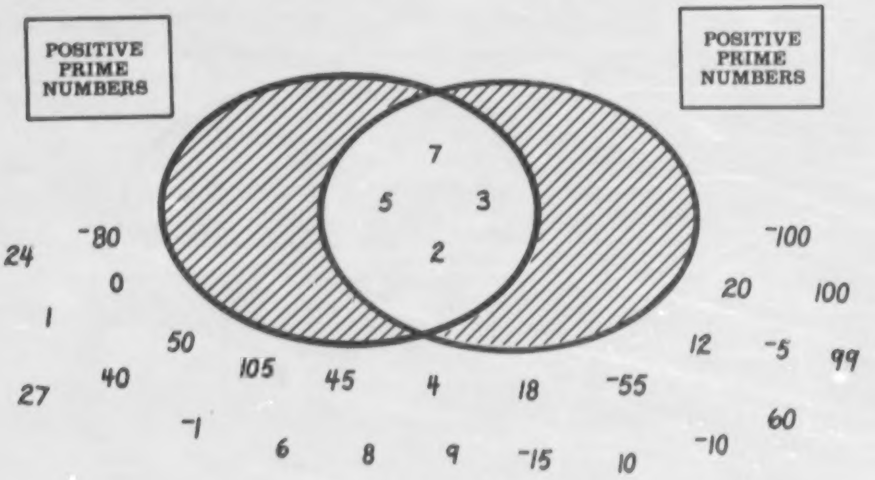
Example 3: One Empty Region


POSITIVE DIVISORS OF 12
LARGER THAN ⁻10
12
2
3
4
6
1
9
0
8
⁻1
40
20
⁻5
100
50
105
99
27
45
24
7
60
18
10
5
⁻80
⁻100
⁻10
⁻55
⁻15

Example 4: One Empty Region


MULTIPLES OF 3
POSITIVE DIVISORS OF 20
105
18
9
0
99
45
6
3
⁻15
24
60
27
12
5
1
4
20
10
2
50
⁻10
40
⁻55
⁻5
8
7
⁻1
⁻100
⁻80
100

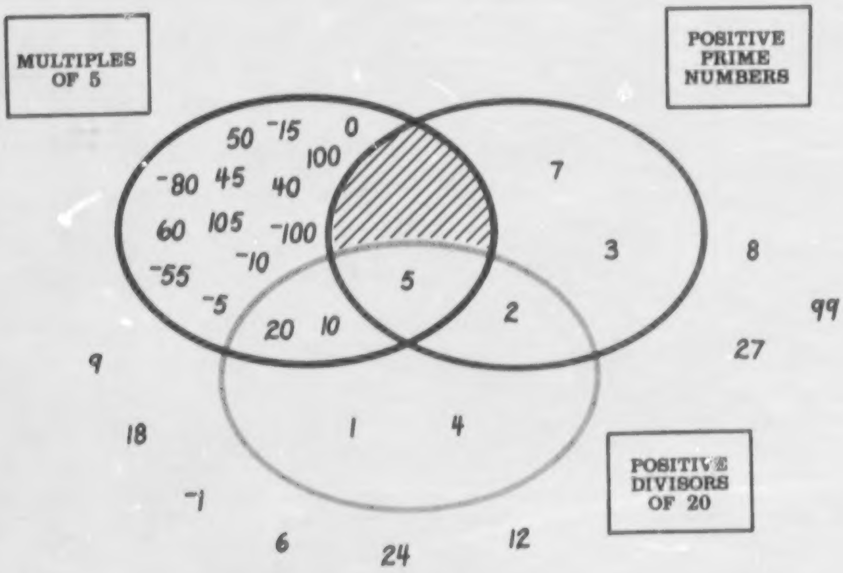
Example 5: Two Empty Regions


POSITIVE PRIME NUMBERS
POSITIVE PRIME NUMBERS
7
5
3
2
24
-80
0
1
50
27
40
105
45
4
18
-55
-1
6
8
9
-15
10
-100
20
100
12
-5
99
60
-10

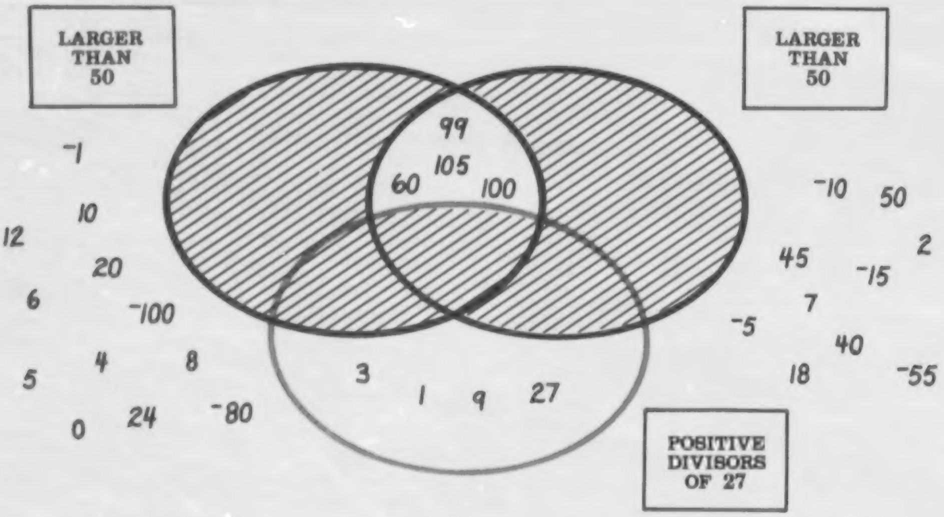
Variation #2: Three Strings

The warning concerning the jump from the two-string variation to the three-string variation on page 138 applies even more so to Version D.

Example 1: One Empty Region


MULTIPLES OF 5
POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 20
50
-15
0
100
-80
45
40
60
105
-100
-10
-55
-5
20
10
5
7
3
2
1
4
8
99
27
9
18
-1
6
24
12

Example 2: Five Empty Regions



Version E: The String Game with Special Scoring Rules

In this version of the game special scoring rules are introduced which determine the winning team. Both the String Game with numbers and with shapes can be played with these scoring rules, however, in the lessons of this book only the Numerical String Game is played this way.

Scoring

Each student is given a list of the possible string labels on a sheet from the individual String Game Analysis Pad. One round of the game is completed by having one player from each team take a turn. After each round the students have an opportunity to privately identify the face-down string cards. They do this by circling on their RED and BLUE lists what they believe is on the hidden tags and submitting it to the judge (teacher). However, each student has just one chance to make a private identification. If a player correctly identifies the string cards after a particular round, then points (according to the following schedule) are added to the player's team score.

Round 1 - 200 points
Round 2 - 150 points
Round 3 - 100 points

Round 4 - 50 points
Round 5 - 25 points
Round 6 - 24 points

Round 7 - 23 points
⋮
Round 29 - 1 point

The team that correctly identifies the string cards at the end of the game scores an additional 100 points plus 5 points for each unplayed game piece the opposing team has left on the Game Board. The winning team is the one with the greater total score.

Comments

1) Keep count of the rounds played on the board so that both you and the students know which round has just been completed.
2) After each round give the students time (perhaps 15-30 seconds) to study the picture, mark their lists, and if desired to submit them.
3) When the students submit their list, write on each one the round after which it was turned in and sort them by teams. You can total the team scores at the end of the game. Do not tell the students that their identifications are correct or incorrect, but encourage them to write their findings on a separate sheet of paper so as not to make mistakes when they play. They should continue to check their identifications during the remainder of the game.
4) It may be even more important to insist on silence during this kind of game, because an individual can have more influence on a team's winning or losing. We suggest you be strict about penalties for talking during the game.

Purpose

Scoring rules should not be imposed on the String Game until students have become very familar and experienced in the play of the game. Many students will already be playing strategically, analyzing the clues given by correctly placed game pieces. Others may still be playing in a random way. The introduction of scoring rules will obviously favor the better students, and perhaps challenge them in a way the usual game no longer is doing. It provides them the opportunity to more dramatically affect their team's chances of winning. For many students the scoring rules may provide incentive to begin analyzing the game and thinking about what information is given by a correctly played game piece.

We do not suggest that you play the String Game very often with the scoring rules. They do impose complication and give perhaps too much advantage to the better players. After playing the game with scoring rules a couple times, you should find that many students have become significantly better players. Returning to the usual way of playing the game, most students will continue to be better players.

One advantage of using scoring rules when you play the String Game and have only a small amount of time is that you can stop the game before the end and have a reasonable way to determine the winning team. The winning team would be the one with the larger total number of points from identification of string cards.

GEOMETRY
AND
MEASUREMENT

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part II

Geometry and Measurement

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR G – LESSONS

WEEK	Geometry and Measurement	Materials
1	ADJUSTMENT DAY	
2	**1G** G1.1: Length #1 p. 5	Teacher: 30 cm ruler Student: 30 cm ruler; colored pencils; Worksheets G01, G02, G03
3	**2G** G1.2: Area #1 p. 10	T: Grid board; colored chalk S: Colored pencils; Worksheets G04, G05, G06
4	**3G** G1.3: Area #2 p. 14	T: Grid board; colored chalk S: Colored pencils; Worksheets G07, G08, G06
5	**1P** P1.1: Making a Fair Choice p. 3	T: Marbles (two different colors); code names (optional); [illegible] (optional) S: None
6	**2P** P1.2: Probabilities p. 21	T: Marbles (two different colors); colored chalk S: Colored pencils; Worksheets P01, P02
7	**4G** G2.1: Parallelism #1 p. 35	T: Demonstration translator; colored chalk S: Translator; large sheets of unlined paper; colored pencils; Worksheets G17, G18
8	**5G** G2.2: Parallelism #2 p. 44	T: Demonstration translator; colored chalk S: Translator; large sheets of unlined paper; colored pencils
9	**6G** G2.3: Parallelism #3 p. 51	T: Demonstration translator; colored chalk; IG-II Geometry Posters #1, #2 S: Translator; colored pencils; large sheets of unlined paper

SUGGESTED SCHEDULE FOR G – LESSONS

WEEK	Geometry and Measurement	Materials
10	**7G** G1.4: Length #2 p. 19	T: 30 cm ruler S: 30 cm ruler; Worksheets G09, G10, G11, G12
11	ADJUSTMENT DAY	
12	**8G** G2.4: Parallelism #4 p. 56	T: Demonstration translator; colored chalk; IG-II Geometry Posters #2, #3, #4 S: Translator; colored pencils; large sheets of unlined paper
13	**9G** G1.5: Area #3 p. 22	T: Grid board; colored chalk S: Colored pencils; Worksheets G13, G14
14	**10G** G2.5: Parallelism #5 p. 62	T: Demonstration translator; colored chalk S: Translator; colored pencils; large sheets of unlined paper
15	**5P** P1.6: Fair Game? #3 p. 50	T: Colored chalk; Teacher's Guide pages 66, 67, and 68. S: None
16	**11G** G2.6: Parallelism #6 p. 67	T: Demonstration translator; colored chalk S: Translator; colored pencils; large sheets of unlined paper
17	**12G** G2.7: Parallelism #7 p. 71	T: Demonstration translator; colored chalk S: Translator; colored pencils; large sheets of unlined paper; Wkshts. G19, G20, G21, G22
18	**13G** G1.6: Area #4 p. 25	T: Grid board; colored chalk S: Colored pencils; Worksheets G15, G16

SUGGESTED CHRONOLOGICAL ORDER OF G -LESSONS

Lesson One G1.1 - Length #1 5
Lesson Two G1.2 - Area #1 10
Lesson Three G1.3 - Area #2 14
Lesson Four G2.1 - Parallelism #1 35
Lesson Five G2.2 - Parallelism #2 44
Lesson Six G2.3 - Parallelism #3 51
Lesson Seven G1.4 - Length #2 19
Lesson Eight G2.4 - Parallelism #4 56
Lesson Nine G1.5 - Area #3 22
Lesson Ten G2.5 - Parallelism #5 62
Lesson Eleven G2.6 - Parallelism #6 67
Lesson Twelve G2.7 - Parallelism #7 71
Lesson Thirteen G1.6 - Area #4 25

GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as number and calculations) and physical concepts (such as length, area, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners". Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill, which often serves to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand provide an opportunity to explore geometric notions informally. The activities are varied to meet other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the students will develop knowledge and skills.

But the effects of such an informal program must be measured in their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

1

MEASUREMENT

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter or for comparison of standard units, the lessons of this chapter provide rich, open-ended experiences which relate measurement to other areas of the mathematics curriculum. The activities enable students to develop congnitive strategies and to build bridges among the strands of the program. Direct experience of the tools and concepts is central. Skills and mastery of measurement tasks should develop naturally.

This chapter comprises three threads: length, area, and spatial relations. Two lessons on length present problems of constructing zigzags under a variety of constraints to encourage the students' developing sense of length measurement and its relation to the number line, as well as to provide practice with the ruler. The lessons on area deal primarily with calculation of area of shapes on a grid, and encourage the principal strategies of dissection and embedding. Begin with simple rectangles and dissect them into constituent squares on a grid. Then consider the area of triangles on a grid by embedding them into rectangles. Finally, dissect and embed irregular shapes into the three regular shapes, squares, rectangles, and triangles. A bonus of such an approach to area problems is the development of the students' sense of spatial relations between parts and the whole.

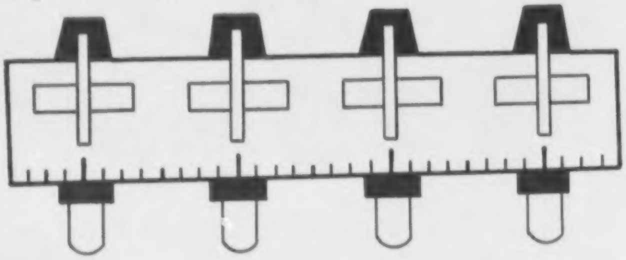
The lessons in this chapter are supplemented regularly with pages in the workbook series, A Set of Problems, No. 1, 2 and 3.

NOTE ON GRIDS

In four lessons in this chapter, as well as many lessons in the other parts of the CSMP Intermediate Grades Program, you will need a large grid board for demonstrations. A portable chalkboard with a permanent grid is convenient and efficient. Squares with eight centimeter sides are ideal. However, if such a grid board is not available to you, there are several alternatives.

Printed grids on large sheets of paper* are provided with full classroom sets of materials for CSMP Mathematics for the Intermediate Grades, Part II. They can be taped to the board and discarded after use. Or one could be mounted on a board and covered with clear contact paper to provide a protective surface. Erasable wax crayon can then be used for drawings.

Another possibility is to carefully draw a grid on the chalkboard each time. The disadvantage of this technique is that inevitably the grid lines are partially erased each time a drawing is altered. However, one can make a tool for drawing grids. Use a music staff drawer with chalk in the first, third, and fifth positions. Or proceed as follows: clip four pen caps to a ruler at eight centimeter intervals and secure them with tape. Pen caps are usually of a diameter to hold a piece of chalk securely.



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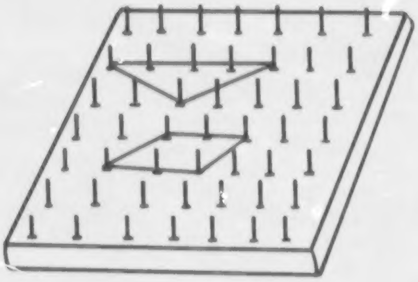
*These printed grids are packaged under the title "Robot Walk Grid". The title is from lessons in the CSMP Kindergarten Program and has nothing to do with the lessons in this chapter.

For lessons in which shapes are to be constructed for area or perimeter calculations, a geoboard often surpasses a grid board in usefulness. A geoboard consists of a grid board with nails or pegs protruding at the corners of the grid.



Instead of drawing lines, one uses rubber bands to construct shapes. Rubber bands are quickly and easily manuevered to facilitate the development of a lesson. In constructing a geoboard, use escutcheon or finishing nails. Let them protrude about two centimeters above the surface of the board. It is not necessary to draw a grid on the geoboard, but sufficient to mark each corner with a spot about one centimeter in diameter. The spots must be visible throughout the room. Omitting the lines of the grid emphasizes the shapes by focusing attention on the rubber bands.

Individual geoboards are compelling tools for the students and provocative in the readiness with which problems can be posed and attacked. Several models are available commercially. They can also be constructed easily. Make a master of a grid for duplication. Squares 3 cm by 3 cm are a good size for individual geoboards of six squares by six squares. Tape a copy of the grid to a piece of one inch plank. Hammer nails at the corners of the grid. Use a pair of pliers as a spacer to ensure uniformity of the nails' heights. Tear off the grid paper. There is no need to paint a grid or spots at the nails as on the demonstration geoboard, since the user is close enough to the geoboard for the nails themselves to define the grid. Use rubber bands of several colors; and for a demonstration geoboard, be sure to have some very large rubber bands.

CAPSULE LESSON SUMMARIES

Section 1.1: Measure line segments with the ruler using centimeters and millimeters. Construct and measure the length of zigzags by measuring and adding the lengths of their segments.
(Worksheets G01, G02, G03)

Section 1.2: Find areas of shapes on a centimeter grid by counting constituent squares. Construct shapes of specified area.
(Worksheets G04, G05, G06)

Section 1.3: Find areas of shapes composed of squares and triangles. Find areas of triangles by enclosing them in rectangles twice as large.
(Worksheets G06, G07, G08)

Section 1.4: Practice using the ruler by solving dot-to-dot puzzles.
(Worksheets G09, G10, G11, G12)

Section 1.5: Find areas of more triangles and complex irregular shapes by decomposing them into simpler shapes.
(Worksheets G13, G14)

Section 1.6: Find areas of parallelograms. Practice spatial perception by extracting parallelograms from complex designs.
(Worksheets G15, G16)

1.1 Length #1

The lesson provides practice in measuring the lengths of line segments in a problem solving setting. Most of the class period is spent in individual effort on worksheets. Each student will need colored pencils and a ruler graduated in centimeters and millimeters.

Begin by reviewing the use of the ruler and the markings on it.

T: What is the distance between two long marks on the ruler?

S: One centimeter.

T: What is the distance between two short marks on the ruler?

S: One millimeter.

T: How many millimeters are there in a centimeter?

S: 10 millimeters in a centimeter.

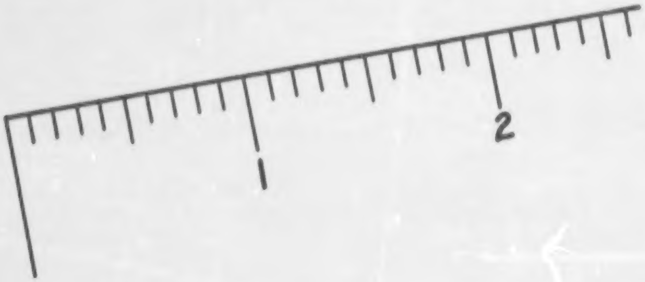
Write this information on the board.

10 mm = 1 cm

T: Remember that we write "mm" for millimeter and "cm" for centimeter.

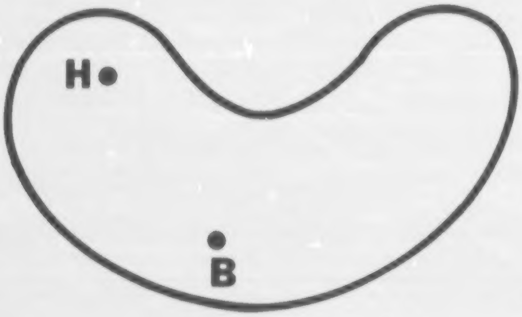
T: Suppose that you measure a line segment and find that its length is 13 centimeters and 8 millimeters. You might write " 13cm 8mm " for the length, but we usually write lengths in a shorter form. We write " 13.8 cm" instead. This is an idea that comes from the way we write amounts of money. Ten dimes equal one dollar and ten millimeters equal one centimeter. 13 dollars and 8 dimes is written "$13.80" or "13.8 dollars", and 13 cm 8 mm is written "13.8 cm".

Draw the following enlargement or use a transparent ruler with an overhead projector.



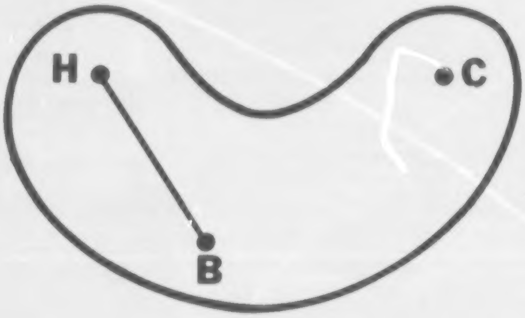
Point in turn to the marks representing 1.3 cm, 1.6 cm, 1.8 cm, and 2.1 cm. Each time ask a student to read the length represented and write that length on the board. Continue until the students have a clear understanding of interpreting the marks.

Draw the following picture on the board.



T: Here is a picture of a school yard. Harry and Bill (the dots H and B) are playing a game in which Harry must run to Bill on as short a path as possible.

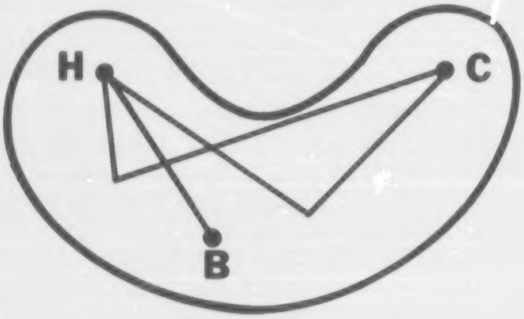
Draw a segment connecting H and B to show the solution. Add point C to your picture.



T: Chester wants to play the same game with Harry, but the children must stay in the school yard, so Chester must run on a zigzag. There are many paths that Chester might take.

Invite students to indicate paths on your drawing. Draw at least two paths from C to H, using a different color for each.

For example,



T: Chester is supposed to run on the shortest path possible. How can we decide which of these paths is better for him?

S: By measuring the lengths of the segments and adding.

Demonstrate the use of the ruler by measuring the lengths of the segments and then perform the additions on the board. To preclude the common error of placing the mark at "1" on the ruler at one end of the line segment, emphasize the correct placement of the end of the ruler (or the mark at "0", if your ruler has one). Distribute Worksheet G01.

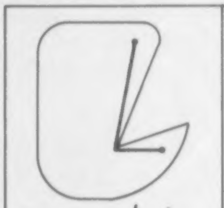
T: Here are four schoolyards. Each has two dots to be connected with a zigzag. Don't go out of the school yard. Measure the lengths of your zigzags and write the lengths in the blanks.

Circulate to monitor the students' use of the ruler in drawing the zigzags (some students might try to draw the segments freehand) and in measuring the lengths of the segments. Encourage them to experiment in seeking zigzags whose lengths are as short as possible. As students finish, distribute Worksheets G02 and G03 in turn.

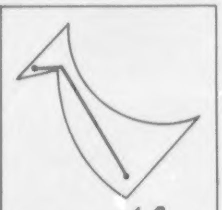
The problems on Worksheet G02 require a strategy to find zigzags whose lengths fit the number stories. There is also some addition practice with decimals. Worksheet G03 is more difficult. Since many zigzags are possible, encourage the students to draw several zigzags and compare their lengths in seeking a shortest path. You may want to spend a few minutes at the end of the lesson having the students compare their results on the worksheets.

Name ______________ G 01 *

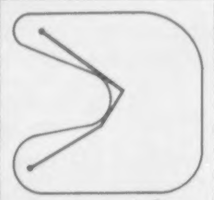
In each picture connect the dots with a zigzag. Do not go out of the school yard. Measure the length of each zigzag. Try to make your zigzag as short as possible.



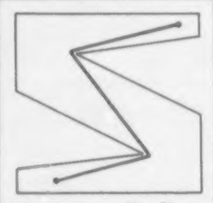
length of zigzag = 4.6 cm



length of zigzag = 6.3 cm



length of zigzag = 9.3 cm

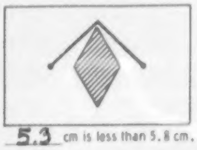


length of zigzag = 14.5 cm

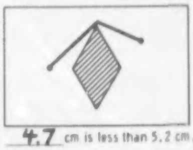
Many acceptable solutions are possible.

Name ______________ G 02 **

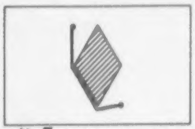
In each picture connect the dots with a zigzag. Do not go out of the school yard or through the building. Measure the length of each zigzag. Try to draw a zigzag which makes the number story true.



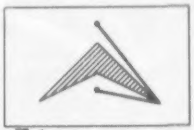
5.3 cm is less than 5.8 cm.



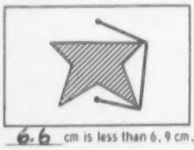
4.7 cm is less than 5.2 cm.



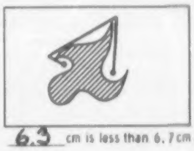
4.5 cm is less than 4.9 cm.



7.1 cm is less than 7.2 cm.



6.6 cm is less than 6.9 cm.

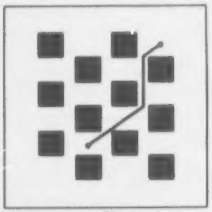


6.3 cm is less than 6.7 cm.

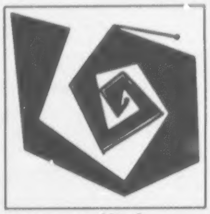
Many acceptable solutions are possible.

In each picture connect the dots with a zigzag. Do not go out of the school yard or through the shaded areas. Make your zigzags as short as possible.

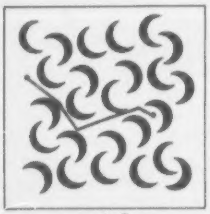
G 03 ***



length = 5.6 cm



length = 17.6 cm



length = 6.3 cm

Make your own problem here.
length = ________ cm

Many solutions are possible.

1.2 Area #1

This is the first of several lessons on area. You will need to use your grid board, geoboard, or grid paper (see the note on grids in the introduction to this chapter) to demonstrate calculating areas of regions by counting squares or parts of squares enclosed in a region. Individual student geoboards would be useful at times in this series of lessons.

Begin by reminding the students of the terms "centimeter" and "millimeter" and the relation "10 mm = 1 cm" from the first measurement lesson. Write all three on the board as you mention them. Have the students read the words a few times.

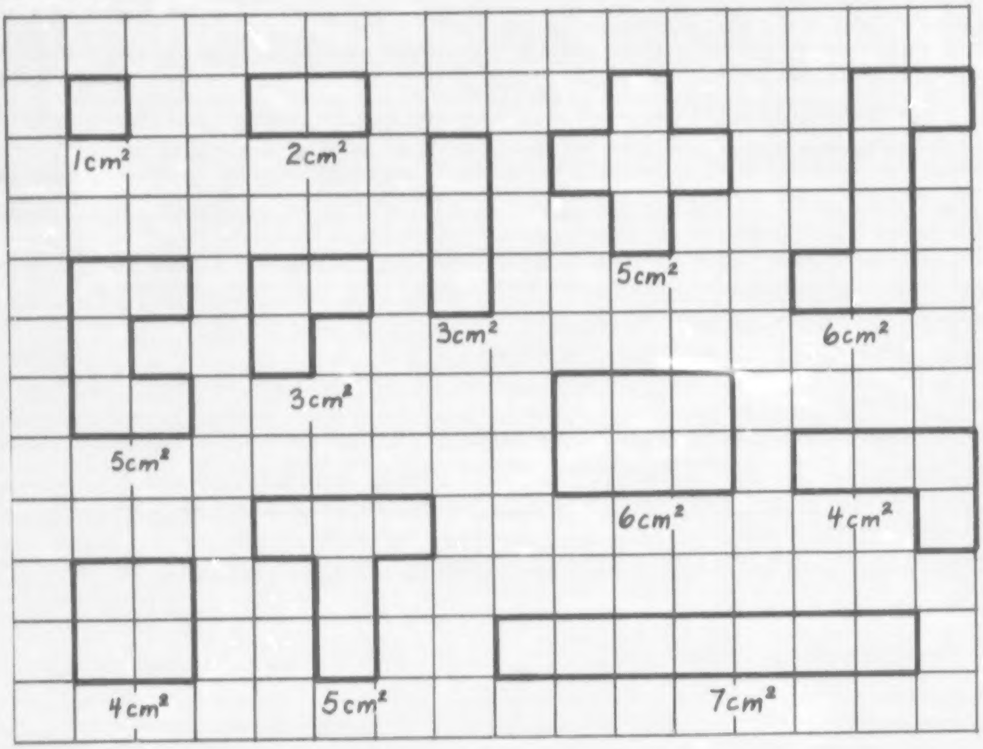
T: Today we will measure the area of shapes that we can draw on the grid board. We will pretend that the length of a side of one of the squares of the grid board is one centimeter. Then the area of a square on the grid board is one square centimeter.

Write this abbreviation on the board.

$$1 \text{ square centimeter} = 1 \text{ cm}^2$$

T: Now I will draw a shape on the grid board and I want you to find its area by counting the number of square centimeters it contains.

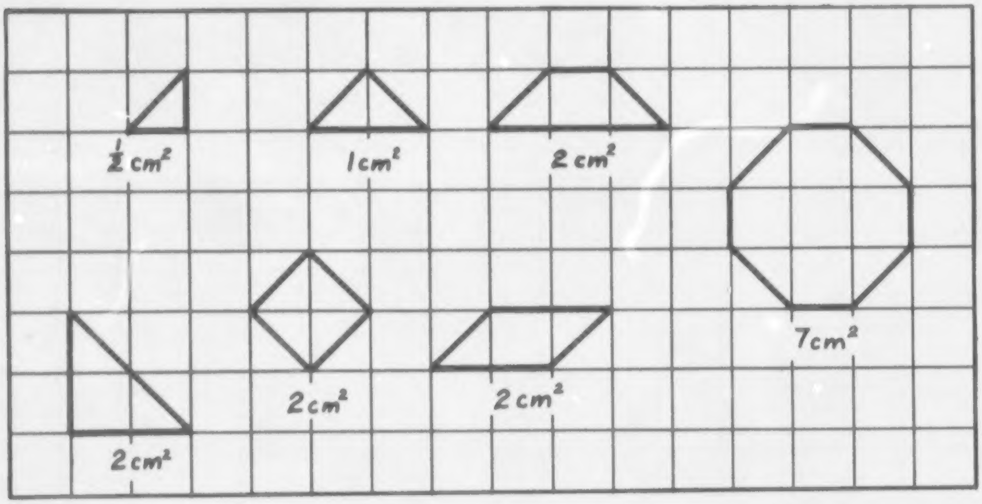
Draw some of the following shapes one at a time. After drawing each shape, encourage discussion among the students to determine its area. If the grid board is large enough to allow the areas to be written neatly on or near the shapes, do so. Otherwise, make a list of the areas on the chalkboard. Do not forget to write the unit "cm²" with the magnitude of each area. Draw enough of the shapes to ensure that most students understand the counting procedure.



T: The perimeter (write this word on the board) of a shape is the length of its border. (Trace the border of one of your shapes.)

Invite students to the grid board to trace the border of several shapes. After a student has traced a border, lead the class in counting the number of centimeters in its length, that is, in determining the perimeter of the shape in centimeters.

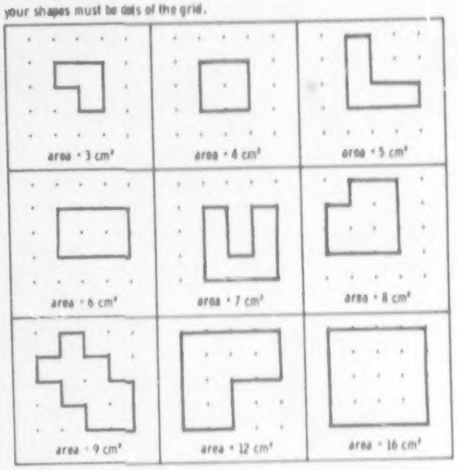
Draw a selection of the shapes from the next illustration and lead the class in finding their areas. Start with the triangular half-square and emphasize that its area is one-half square centimeter. The other shapes can be decomposed into a collection of squares and triangular half-squares to find their areas.



If more collective work is necessary, construct other similar shapes and find their areas. At the end of the collective work, distribute Worksheets G04 and G05 and monitor the students' work. Worksheet G06 provides a supplementary problem for this lesson and Section 1.3.

Name ____________________ G 04 *

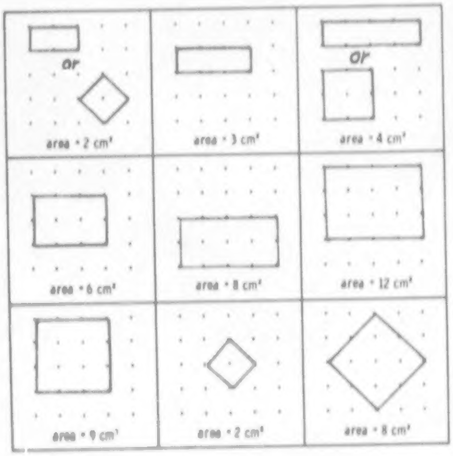
In each grid draw a red shape with the correct area. Rule: the corners of your shapes must be dots of the grid.


area = 3 cm^2
area = 4 cm^2
area = 5 cm^2
area = 6 cm^2
area = 7 cm^2
area = 8 cm^2
area = 9 cm^2
area = 12 cm^2
area = 16 cm^2

A variety of solutions is possible.

Name ____________________ G 05 **

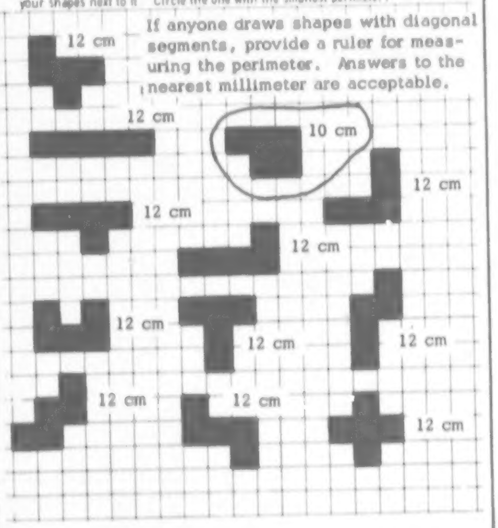
In each grid draw a blue square or a blue rectangle with the correct area.


or
area = 2 cm^2
area = 3 cm^2
or
area = 4 cm^2
area = 6 cm^2
area = 8 cm^2
area = 12 cm^2
area = 9 cm^2
area = 2 cm^2
area = 8 cm^2

Try to make squares with the last three areas.

Name ____________________ G 06 ***

Draw as many red shapes as you can with area 5 cm^2. The perimeter of a shape is the length of its border. Find and write the perimeter of each of your shapes next to it. Circle the one with the smallest perimeter.


12 cm
12 cm
10 cm
12 cm
12 cm
12 cm
12 cm
12 cm
12 cm
12 cm
12 cm
12 cm
12 cm

If anyone draws shapes with diagonal segments, provide a ruler for measuring the perimeter. Answers to the nearest millimeter are acceptable.

A variety of shapes is possible.

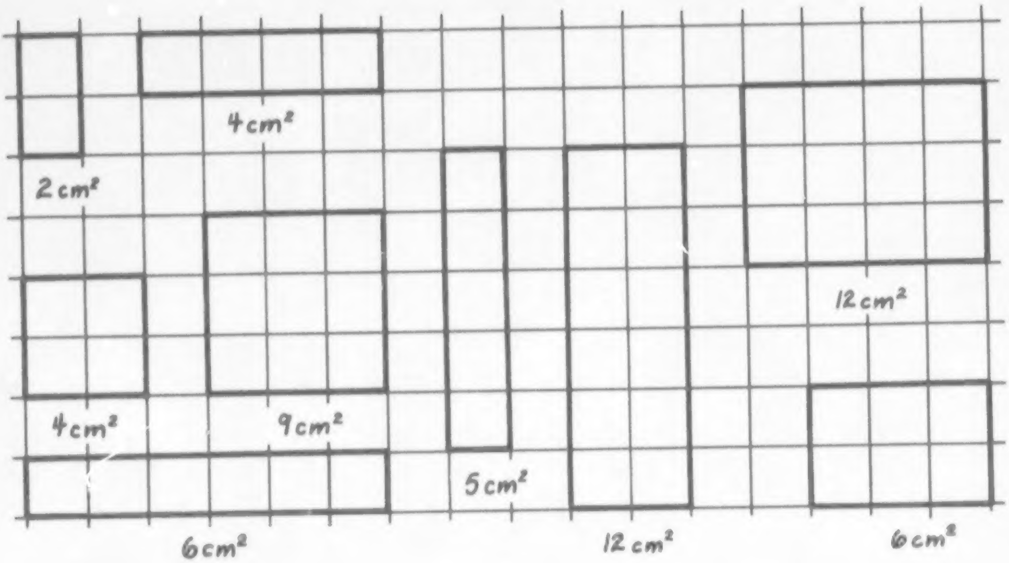
1.3 Area #2

Continuing the work with area, this lesson introduces fractional areas and areas of more triangular shapes. Use your large grid board, geoboard, or grid paper (see the note on grids in the introduction to this chapter) to present shapes for discussions.

Begin by reviewing the terms "centimeter" and "millimeter" and the relation "10 mm = 1 cm". Write all three on the chalkboard as you mention them and have the students read each a few times. Review also the term "square centimeter" from Section 1.2, writing "cm^2" on the board.

T: We will measure areas of shapes today and again we will pretend that the area of each of the squares of the grid board is one square centimeter.

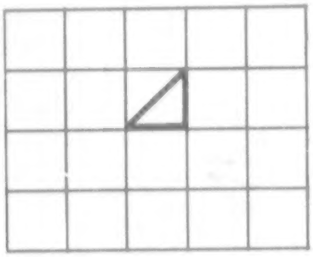
Construct several rectangles on the grid board. After you draw a rectangle, ask the students to calculate its area. Write the area on the grid board itself, if convenient, or in a list on the chalkboard. In this collective exercise try to include rectangles whose areas range up to 12 cm^2 and, for areas with more than one representation, present more than one rectangle. For example, represent 6 cm^2 by rectangles of sizes 1 cm by 6 cm and 2 cm by 3 cm, and represent 12 cm^2 by rectangles of sizes 2 cm by 6 cm and 3 cm by 4 cm. The next illustration shows a good collection of rectangles for this exercise.

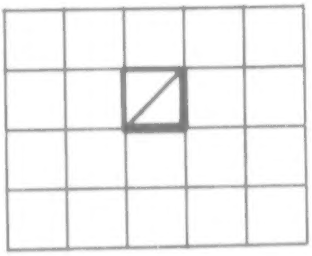


After drawing several of these rectangles, reverse the problem and invite students to trace rectangles on the grid board whose areas are given by you. Suggest areas ranging up to 12 cm³.

T: So far we have been looking only at rectangles. Let's look at some other shapes, for example a triangle.

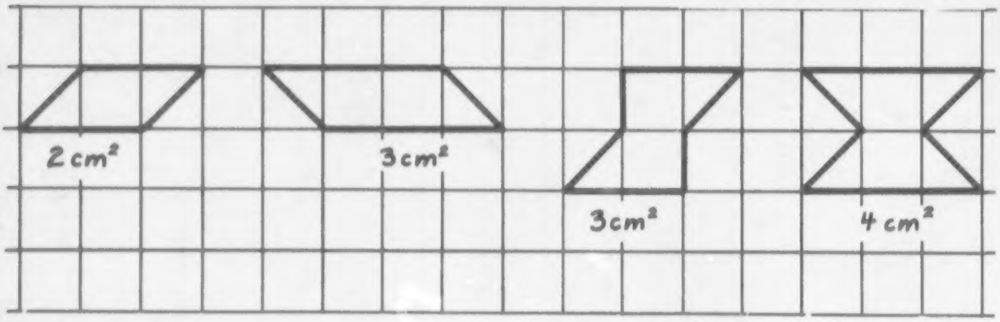
Draw the triangle at the left below and lead a discussion of finding its area. A good method is to enclose the triangle in a square and observe that the triangle is one-half of the square (that is, its area is one-half that of the square). If no student suggests this method, do so yourself. Ask a student to trace or draw the square on the grid board.



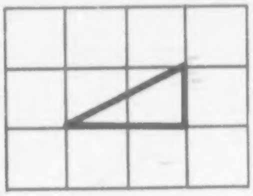


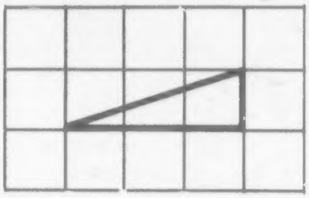
3G

Proceed with the following shapes. Ask the students to find their areas by noting the number of squares and triangles which they enclose. Invite students to the board to demonstrate the calculations.

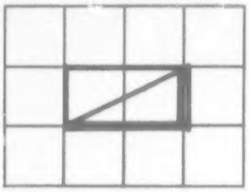


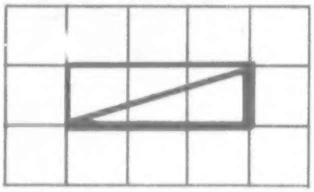
Next present two more triangles in turn.





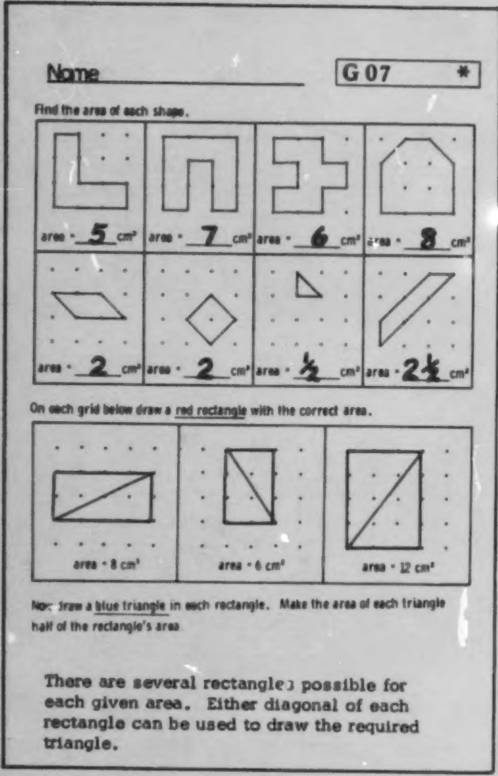
Assist the students to find the areas of these triangles by enclosing them in rectangles and taking one-half of the rectangles' areas. Ask students to demonstrate the procedure at the grid board.





Distribute Worksheets G07 and G08 and monitor the students' work. Pay special attention to their counting techniques in finding areas. Encourage students who finish both of these worksheets to work again on G06.

NOTE: In this lesson you will have occasion to use the terms "triangle" and "parallelogram" in addition to "rectangle" and "square". Use them naturally without particular emphasis. If a student shows any confusion in using one of these terms, simply offer a correction. More work with parallelograms will follow, so a casual introduction to the word is valuable. Most students are familiar with triangles, rectangles, and squares, although some may confuse rectangles and triangles. The confusion is usually only terminological and rectified by comments such as "Triangles have only three sides" or "How many sides does a triangle have?" A prevalent confusion lies in not recognizing that every square is also a rectangle, but not vice versa. Occasional notice of this fact will be valuable. Remember that a square is a rectangle all of whose sides are equal in length.



Name

G 07 *

Find the area of each shape.

area = 5 cm^2 | area = 7 cm^2 | area = 6 cm^2 | area = 8 cm^2

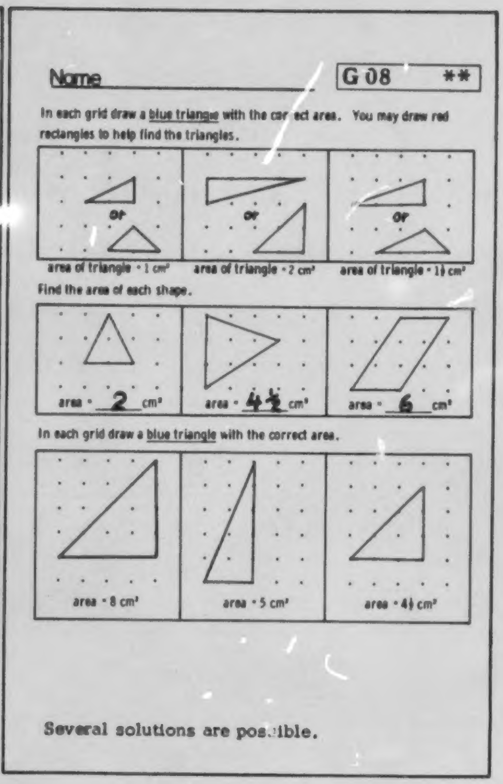
area = 2 cm^2 | area = 2 cm^2 | area = $\frac{1}{2}$ cm^2 | area = 2$\frac{1}{2}$ cm^2

On each grid below draw a red rectangle with the correct area.

area = 8 cm^2 | area = 6 cm^2 | area = 12 cm^2

Now draw a blue triangle in each rectangle. Make the area of each triangle half of the rectangle's area.

There are several rectangles possible for each given area. Either diagonal of each rectangle can be used to draw the required triangle.



Name

G 08 **

In each grid draw a blue triangle with the correct area. You may draw red rectangles to help find the triangles.

area of triangle = 1 cm^2 | area of triangle = 2 cm^2 | area of triangle = 1$\frac{1}{2}$ cm^2

Find the area of each shape.

area = 2 cm^2 | area = 4$\frac{1}{2}$ cm^2 | area = 6 cm^2

In each grid draw a blue triangle with the correct area.

area = 8 cm^2 | area = 5 cm^2 | area = 4$\frac{1}{2}$ cm^2

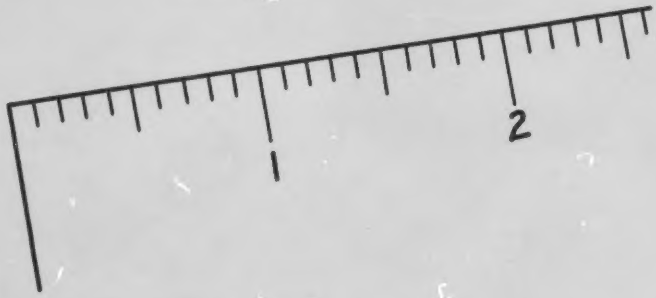
Several solutions are possible.

1.4 Length #2

This lesson offers further practice with a ruler in measuring the lengths of line segments. Each student will need colored pencils and a ruler graduated in centimeters and millimeters. Begin with a quick review as in Section 1.1 of the use of the ruler, its markings and the terms "centimeter" and "millimeter" Emphasize the relation "10 mm = 1 cm". Write all of this information on the board and have the children read the statements. Review the use of the decimal point in writing lengths as follows.

T: Suppose that you measure a line segment and find that its length is 14 centimeters and 6 millimeters. You might write "14 cm 6 mm" for the length, but we usually write lengths in a shorter form. We write "14.6 cm" instead. This is an idea that comes from the way we write amounts of money. Ten dimes equal one dollar and ten millimeters equal one centimeter. 14 dollars and 6 dimes is written "$14.60" or "14.6 dollars", and 14 cm 6 mm is written "14.6 cm".

Draw the following enlargement or use a transparent ruler with an overhead projector.



Point in turn to the marks representing 1.7 cm, 1.5 cm, 0.8 cm, and 2.3 cm. Each time ask a student to read the length represented and write that length on the board. Continue until the students have a clear understanding of interpreting the marks.

Draw a line segment on the board (or overhead projector).

Demonstrate the measurement of its length. To preclude the common error of placing the mark at "1" on the ruler at one end of the line segment, emphasize the correct placement of the end of the ruler (or the mark at "0", if your ruler has one). Next draw a zigzag and demonstrate finding its length to illustrate proper alignment of decimal points in a summation.

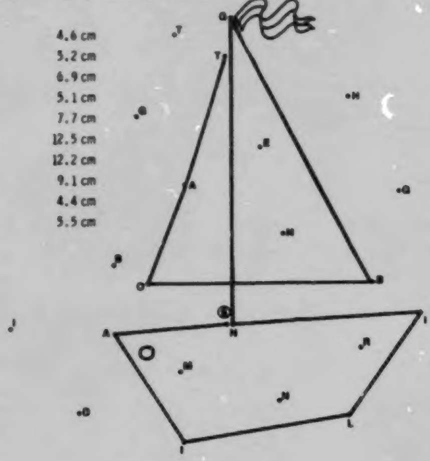
The four worksheets provide practice in measuring lengths in centimeters and millimeters. On each page the student constructs a zigzag starting at the circled letter. The lengths of the successive segments of the zigzag are listed to enable the student to find the sequence of dots to be used. Listing, in order, the letters which appear with the dots of the zigzag gives a message related to the picture. No dot is used twice.

After you have distributed Worksheets G09 and G10, discuss the instructions with the class. Worksheets G11 and G12 are available for those who finish or for later use.

Name ____________ **G 09 ***

Follow the dots. Draw a zigzag starting at the dot marked Ⓢ.
Use this list of lengths for the segments of the zigzag.

4.6 cm
5.2 cm
6.9 cm
5.1 cm
7.7 cm
12.5 cm
12.2 cm
9.1 cm
4.4 cm
5.5 cm

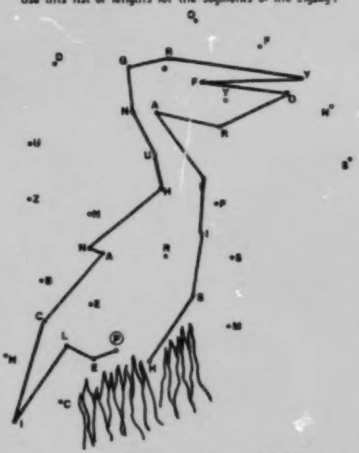


SAILING BOAT

Name ____________ **G 10 ****

Follow the dots. Draw a zigzag starting at the dot marked Ⓟ.
Use this list of lengths for the segments of the zigzag.

1.0 cm
1.2 cm
3.8 cm
4.3 cm
3.7 cm
0.6 cm
3.9 cm
1.4 cm
2.0 cm
1.8 cm
1.7 cm
5.5 cm
4.1 cm
3.5 cm
3.1 cm
2.7 cm
3.3 cm
2.3 cm
2.6 cm
3.3 cm

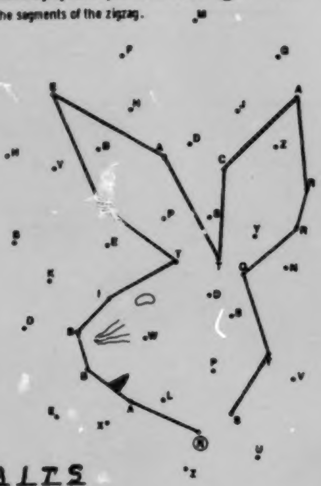


PELICAN HUNGRY
FOR A FISH

Name ____________ **G 11 *****

Follow the dots. Draw a zigzag starting at the dot marked Ⓡ. Use this list of lengths for the segments of the zigzag.

3.1 cm
2.2 cm
1.6 cm
1.9 cm
3.1 cm
3.9 cm
4.9 cm
5.2 cm
4.8 cm
3.7 cm
4.2 cm
3.8 cm
1.7 cm
2.9 cm
3.4 cm
2.9 cm

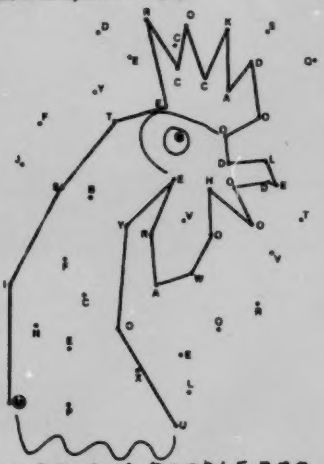


RABBITS
EAT
CARROTS

Name ____________ **G 12 ******

Follow the dots. Draw a zigzag starting at the dot marked Ⓜ.
Use this list of lengths for the segments of the zigzag.

4.9 cm
4.5 cm
3.5 cm
2.2 cm
3.7 cm
2.4 cm
1.8 cm
2.3 cm
2.2 cm
2.5 cm
1.5 cm
2.3 cm
1.6 cm
1.1 cm
1.5 cm
1.1 cm
0.3 cm
1.4 cm
1.7 cm
2.2 cm
1.9 cm
1.8 cm
1.6 cm
2.1 cm
2.5 cm
2.8 cm
4.2 cm
4.7 cm

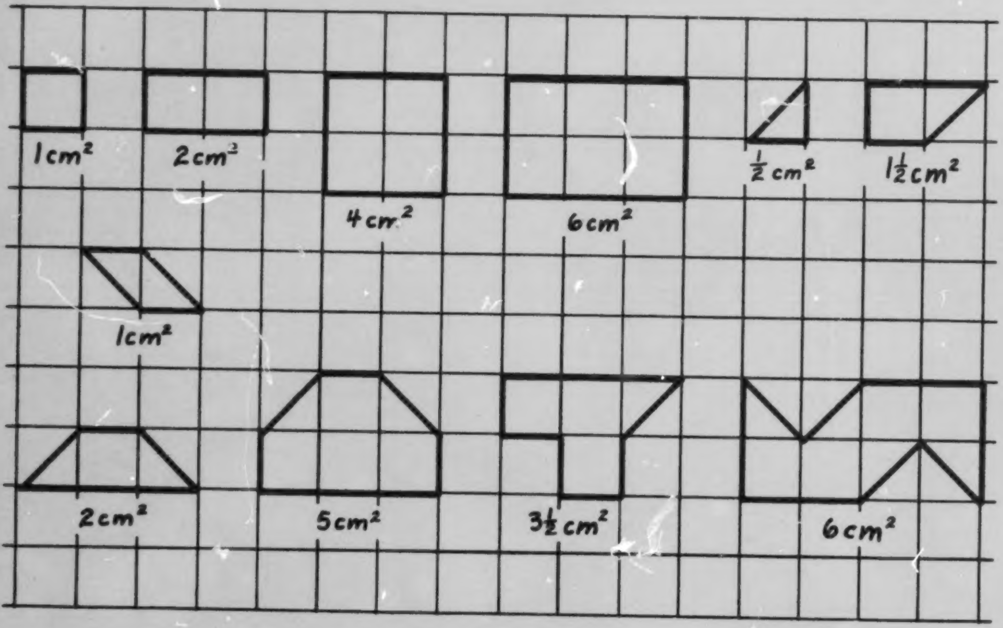


MISTER COCK-A-DOODLE DOO,
HOW ARE YOU?

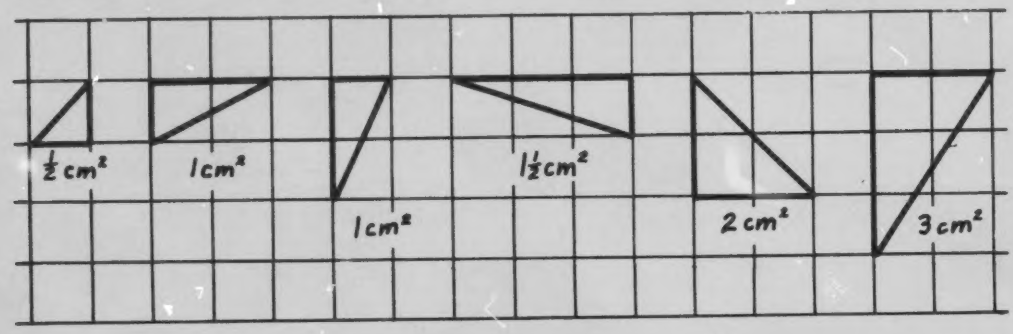
1.5 Area #3

This lesson continues the development of techniques in finding areas of shapes on a grid by counting constituent squares and triangles. Use your grid board, geoboard, or grid paper (see the note on grids in the introduction to this chapter) to present shapes for analysis.

Begin by drawing rectangles and then gradually increase the complexity of the shapes. For each shape, ask a student to demonstrate the calculation of its area at the grid board. One possible sequence of shapes you might use is shown below.

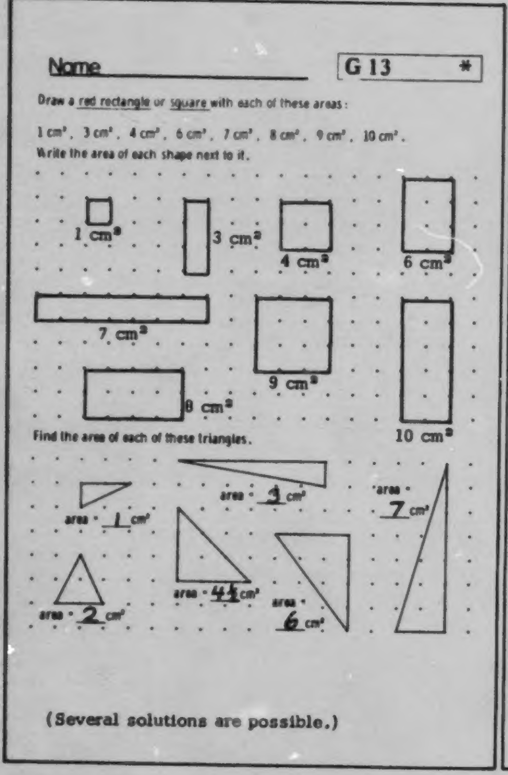


Recall the calculation of the area of a triangle on the grid board by enclosing it in a rectangle, finding the area of the rectangle, and taking one-half of the rectangle's area (or dividing that area by two). For each of the following triangles, ask a student to demonstrate the calculation at the grid board.



Distribute Worksheets G13 and G14. As an additional exercise for students who finish the two worksheets, provide them with centimeter grid paper and ask them to construct triangles of the following areas: $\frac{1}{2}$ cm^2, 1 cm^2, $1\frac{1}{2}$ cm^2, 2 cm^2, $2\frac{1}{2}$ cm^2, 3 cm^2, $3\frac{1}{2}$ cm^2, 4 cm^2, $4\frac{1}{2}$ cm^2, and 5 cm^2.

NOTE: Worksheets G01 and G06 have centimeter grids printed on their reverse sides. There is also a sheet of grid paper with a centimeter grid on both sides between Worksheets L22 and G01 in the Worksheet booklet.



Name

G 13 *

Draw a red rectangle or square with each of these areas:

1 cm², 3 cm², 4 cm², 6 cm², 7 cm², 8 cm², 9 cm², 10 cm².

Write the area of each shape next to it.

Find the area of each of these triangles.

area = 1 cm²

area = 3 cm²

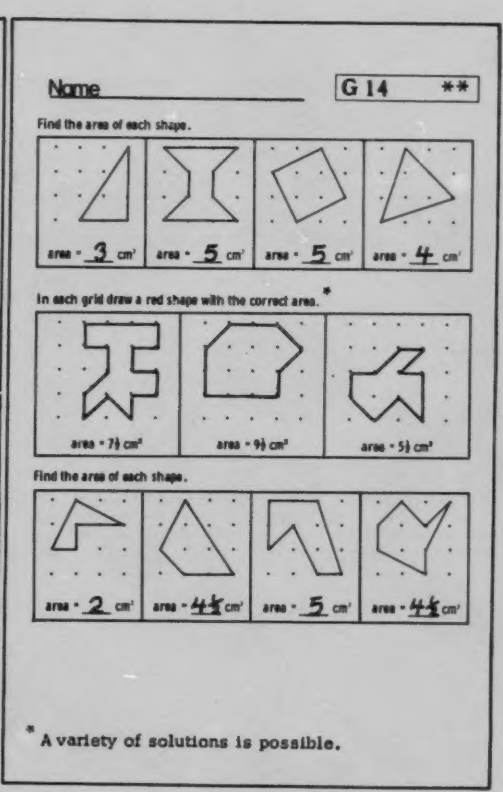
area = 7 cm²

area = 4½ cm²

area = 6 cm²

area = 2 cm²

(Several solutions are possible.)



Name

G 14 **

Find the area of each shape.

area = 3 cm²

area = 5 cm²

area = 5 cm²

area = 4 cm²

In each grid draw a red shape with the correct area.*

area = 7½ cm²

area = 9½ cm²

area = 5½ cm²

Find the area of each shape.

area = 2 cm²

area = 4½ cm²

area = 5 cm²

area = 4½ cm²

* A variety of solutions is possible.

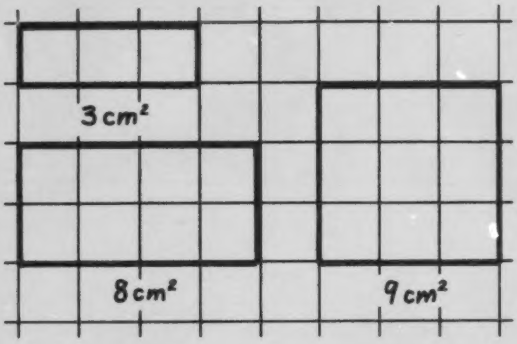
1.6 Area #4

This lesson has two components: practice in calculating areas of shapes on the grid board, and practice in recognizing parallelograms embedded in more complicated shapes. Use your grid board, geoboard, or grid paper (see the note on grids in the introduction to this chapter) to present shapes.

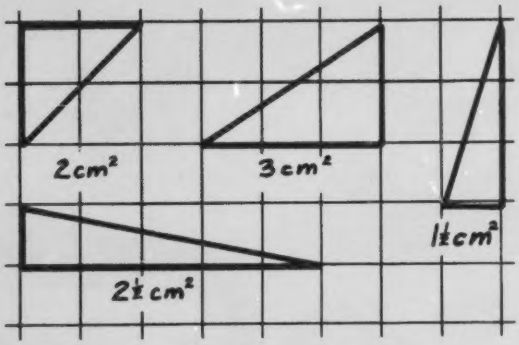
Begin by drawing a variety of triangles, squares, rectangles, and parallelograms and asking the students to identify them. Emphasize the fact that every square is also a rectangle and every rectangle is also a parallelogram.

Present several shapes on the grid board. Ask the students to find their areas. In each case, ask a student to demonstrate the calculation at the grid board.

Begin with a few rectangles, including squares.

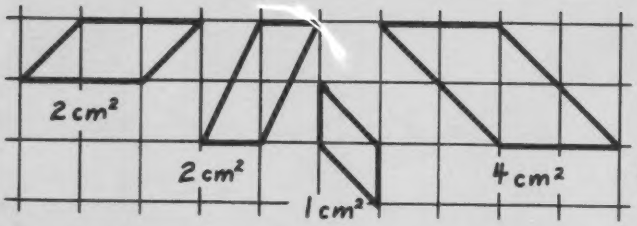


Continue with a few triangles, including some with fractional areas.



13 G

Then, present four or five parallelograms such as the ones pictured at the right.



Draw Figure 1 on the chalkboard. Use a demonstration translator to draw it accurately. Ask the students to find all of the parallelograms in this figure. There are three, the two which are juxtaposed to create the figure, and one which can be seen by ignoring the joining line segment as suggested in Figure 2. Lead discussion analyzing the figure and asking students to trace parallelograms as they are identified.

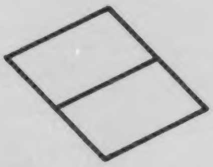


Figure 1

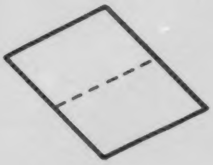


Figure 2

Repeat the exercise with Figure 3 to find ten parallelograms. There are five "singles" which comprise the figure. There are four "doubles" (similar to Figure 2) and one "triple" as shown in Figure 4.

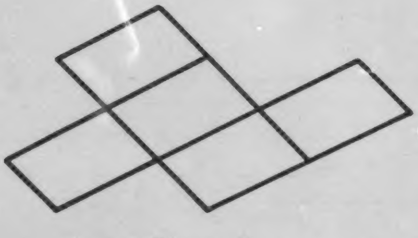


Figure 3

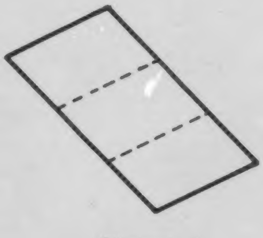


Figure 4

Finish the collective part of the lesson with the following study in pattern recognition. Draw Figure A on the board, preparing to draw a column of similar figures. Write "1" to the left of the parallelogram. Tell the students that you will draw some figures for counting parallelograms. Draw Figure B and ask the class again to count the number of parallelograms in the figure. There are three. Write "3" to the left of the figure and draw Figure C below it. Ask the class to find all the parallelograms in the new figure. Invite students to trace parallelograms as they are discovered. Write "6" to the left of Figure C and below the "3". Now draw Figure D by first repeating Figure C and then adding one more parallelogram to the right end. Discuss this relation between Figures D and C as you are drawing. Comment that the six parallelograms which were found in Figure C are similarly part of Figure D. Trace them. Then ask how many more parallelograms result from the addition of the extra piece in constructing Figure D. Invite students to trace these parallelograms on your drawing until all four have been found. Write "10" to the left of Figure D. See the illustration following this paragraph for an analysis of Figure D. Continue by constructing Figure E from a copy of Figure D, observing that ten parallelograms are in the part of Figure E which is a copy of Figure D, and finding five more

1

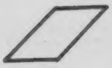


Figure A

3

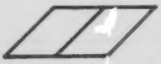


Figure B

6

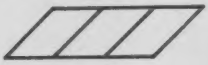


Figure C

10

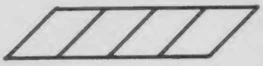


Figure D

15

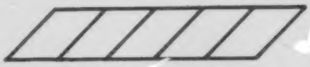


Figure E

21

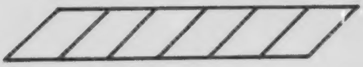
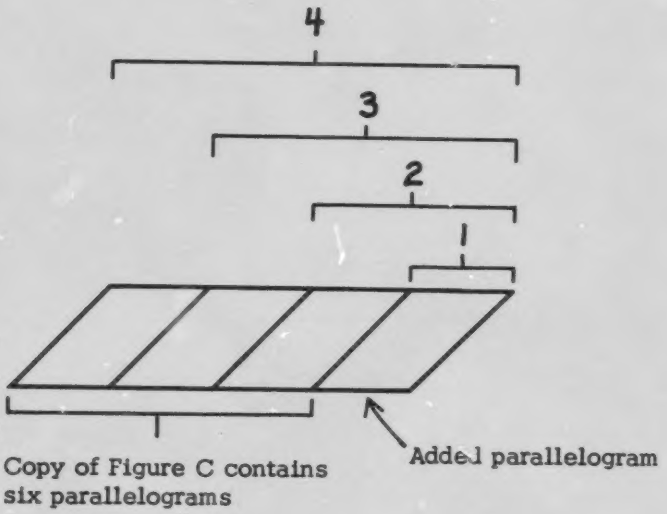
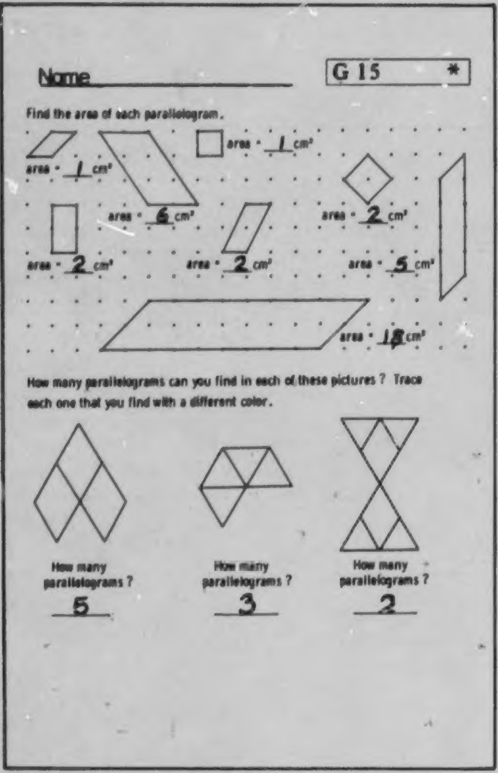


Figure F

parallelograms arising from the additional piece. Write "15" to the left of Figure E. Repeat the process with Figure F. Encourage discussion of the sequence of integers arising in this problem, the way in which each number in the sequence comes from the previous entries and the number of cells in the corresponding figure. Note that the infinite sequence which comes from the iteration of this construction begins: 1, 3, 6, 10, 15, 21, 28, 36, 45. Compare this with the sequence in Exercise 1 of Section 5.2 in the World of Numbers strand.



Distribute Worksheets G15 and G16 and monitor the students' work. Pay close attention to their techniques of area calculation. Most should have developed a good and effective system by this lesson. The problem of counting parallelograms in more complicated figures will be new to most students. At the end of the lesson have students present some solutions, drawing copies of the figures from the worksheets on the board.



Name

G 15 *

Find the area of each parallelogram.

area = 1 cm²

area = 1 cm²

area = 6 cm²

area = 2 cm²

area = 2 cm²

area = 2 cm²

area = 5 cm²

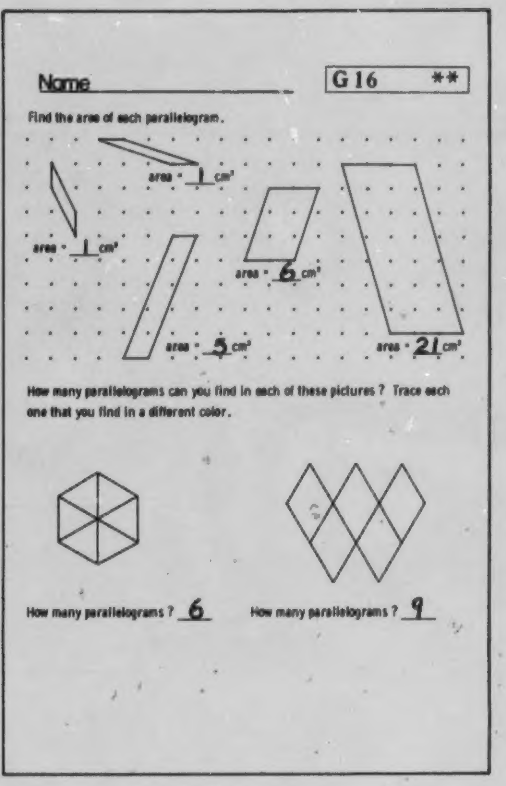
area = 18 cm²

How many parallelograms can you find in each of these pictures? Trace each one that you find with a different color.

How many parallelograms? 5

How many parallelograms? 3

How many parallelograms? 2



Name

G 16 **

Find the area of each parallelogram.

area = 1 cm²

area = 1 cm²

area = 6 cm²

area = 5 cm²

area = 21 cm²

How many parallelograms can you find in each of these pictures? Trace each one that you find in a different color.

How many parallelograms? 6

How many parallelograms? 9

2

PARALLELISM

Suggestions of parallel lines are everywhere: railroad tracks across the Great Plains, the glide of a mated pair of mallards, the opposite sides of a desk, the ruled lines on notebook paper. The architectural embodiments of parallelism indicate its role even in the earliest geometry. From its formal debut as a central concept of Euclid's Elements, a rich and fascinating study of parallel lines and related objects had developed over many centuries.

Although traditionally postponed to secondary school, the notions of parallelism appeal directly to one's intuition. Equipped with a rolling straightedge called a "translator" and a compelling dynamic definition of parallel line segments, students of the lessons in this chapter will perform free and open, but accurate constructions based on parallel line segments and parallelograms. The lessons provide an opportunity to develop familiarity with the translator in constructing families of parallel line segments as well as to begin to understand the geometric relation of parallelism. The first lesson introduces the basic constructions of the chapter, that of the parallelogram and the parallelogram train. These constructions appear in each lesson and provide the principal problem solving tools.

Equipollence, a relation on couples of points, is a natural adjunct to the parallelogram construction. Roughly speaking, two couples of points are equipollent if they define opposite sides of a parallelogram. Several lessons focus on constructing families of equipollent couples of points under a variety of constraints. A colored arrow connecting the points of a couple is used to emphasize the relationships; equipollent couples are indicated by the same color arrows. In fact, the definition of equipollence is made for ordered pairs, the order represented by the arrow's direction. Drawing the arrows as (directed) line segments allows their use as sides of parallelograms. Thus we have a visually stimulating representation for equipollent families of couples.

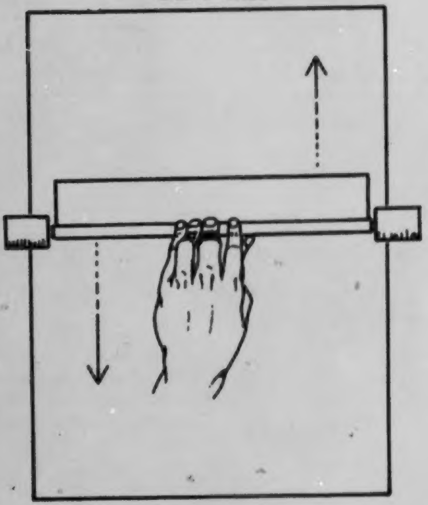
These lessons provide an overt connection with the World of Numbers strand. The equipollence relation together with a properly directed parallelogram train construction yields a number line. Constructing number lines and locating points on them foster the student's developing understanding of the interrelations among numbers. A connection to the language strand is also made in that equipollent families define a function on the plane whose arrow representation is exactly that shown by the arrows of the family.

Parallelism and equipollence will appear regularly in subsequent parts of CSMP Mathematics for the Intermediate Grades. Hesitation and unease at the unusual character of the lessons should be treated by remembering the spirit of the spiral curriculum and of development of understanding through experience. We reiterate: no goals for mastery can be posted here.... The focus of this strand is experience.

NOTE ON TRANSLATORS

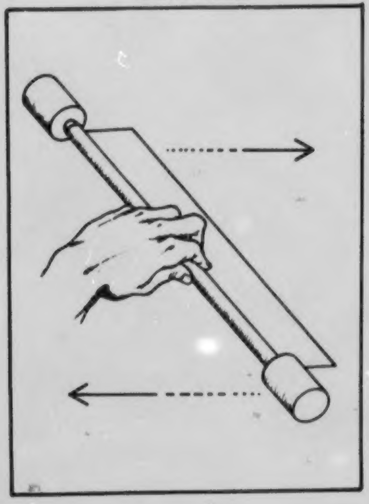
Before using the translator in the classroom, practice using it yourself. Take some time to practice using a student translator as well as the demonstration translator. Occasionally a construction at the board requires an odd posture. In introducing the class to the translator, mention the sharp corners of the metal blade as a warning against misuse. Provide a few small pieces of masking tape to each student so that paper can be anchored to the desk. This precaution will eliminate frustrating slipping of the paper while a student is drawing. Instruct the class in the proper manipulation of the translator. One should hold the plastic tube at the center and gently push or pull to move it across the paper or desk. The following pictures illustrate proper and improper handling.

Like this

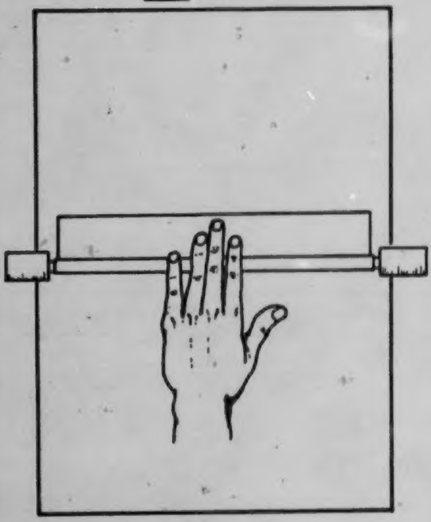


or

Like this

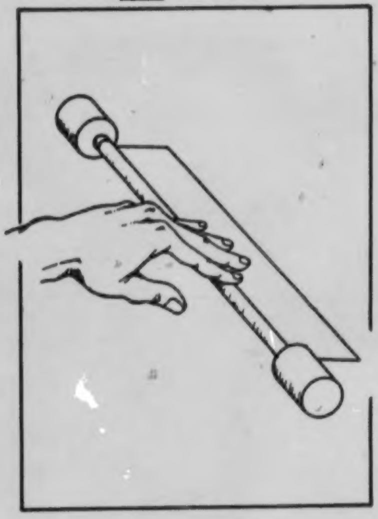


Not like this



and

Not like this



That the blade is not rigidly attached to the plastic tube should cause no problem when a slow steady pressure is used in rolling the translator. The advantage of this means of attachment is that blades may be easily repaired in the event of breakage. For repairs, simply provide yourself with a roll of strapping tape as used in the construction of the translator. However, we anticipate that repairs will only rarely be necessary.

CAPSULE LESSON SUMMARIES

Section 2.1:	Introduce the translator and the basic parallelogram construction. Practice using the translator and draw several parallelogram trains. (Worksheets G17, G18)
Section 2.2:	Draw parallelogram trains. Introduce couples and equipollence. Draw more trains to construct families of equipollent couples.
Section 2.3:	Practice recognizing equipollent couples. Construct families of equipollent couples by drawing parallelogram trains. (Geometry Posters #1, #2)
Section 2.4:	Practice recognizing equipollent couples. Construct families of equipollent couples with specified starting and ending points. (Geometry Posters #2, #3, #4)
Section 2.5:	Construct a family of equipollent couples with carefully prescribed starting points to achieve a number line. Solve problems of labeling points on the number line.
Section 2.6:	Again construct families of equipollent couples with specified starting and ending points.
Section 2.7:	Recognize the pairs of equipollent couples in a parallelogram. Repeat the number line construction of Section 2.5 and practice labeling points on it. (Worksheets G19, G20, G21, G22)

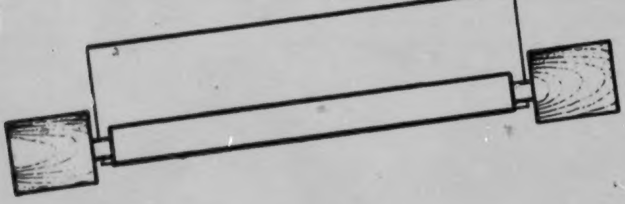
2.1 Parallelism #1

This lesson introduces the translator and the basic parallelogram construction. Familiarity with the concept of parallel lines and dexterity with the translator in performing the constructions will grow over the course of the sequence of lessons in this chapter. Do not be alarmed if at first the children have difficulty using the translator.

An important distinction for you to bear in mind during these lessons is that between lines and line segments. While you will probably use whatever terminology is customary with you, the necessity for making the distinction may arise. Your students would benefit from learning the distinction and we have been faithful to it in writing these lessons. In presenting a lesson draw long line segments, except when drawing specific parallelograms where the sizes of the segments are determined. Occasionally indicate to the class that a segment is only part of a line and that the line can be imagined by extending both ends of the line segment "to infinity" or "forever".

Read the section on usage of the translator in the introduction to this chapter, and practice drawing some line segments and some parallelograms on the board and on paper before you teach this lesson.

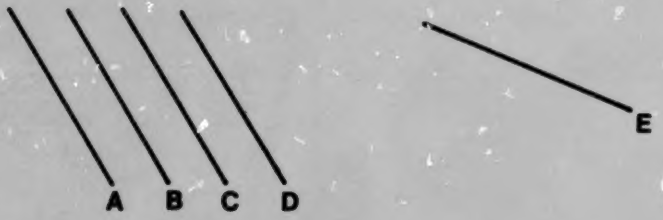
Display your demonstration translator.



4 G

T: This is a translator. We use it to draw line segments.

Draw a long line segment (A in the following illustration) on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The translator has wheels so that it will roll across the board or across your paper. (Demonstrate rolling the translator.) Rolling the translator allows us to draw many, many line segments.

Starting at your original segment and rolling the translator to new positions, draw several more segments (B, C, D) each time rolling the translator 15-20 centimeters.

T: We say that these line segments are parallel. (Write the word on the board.) Two lines are parallel if the translator can be aligned with one of them and then rolled to be aligned with the other.

Demonstrate alignment and rolling with several pairs of the line segments which you have already drawn. Then draw a segment (E) at an angle to those already drawn.

T: Is this line segment parallel to the others? (No)

Demonstrate that alignment along one of the line segments (A, B, C or D) and rolling results in the line of the translator crossing the last line segment drawn (E).

T: Two lines are not parallel if the translator can be aligned with one and then rolled so that the line of the translator crosses the other.

NOTE: In our development classes we used a chant to accompany and to reinforce the use of the translator.

"Parallel, parallel, parallel, roll"

The last word is emphasized and extended. W[illegible]sed the chant a few times at the beginning of each class as a warm-up and stimulus to remember the dynamics of the translator.

T: We can also use the translator to draw a special shape.

The construction is easier to perform than to describe. Describe your actions while performing the construction and emphasize the verbs.

T: Watch me draw a shape with the translator.

A) I draw a line segment ① . (The numbers refer to the next illustration.)

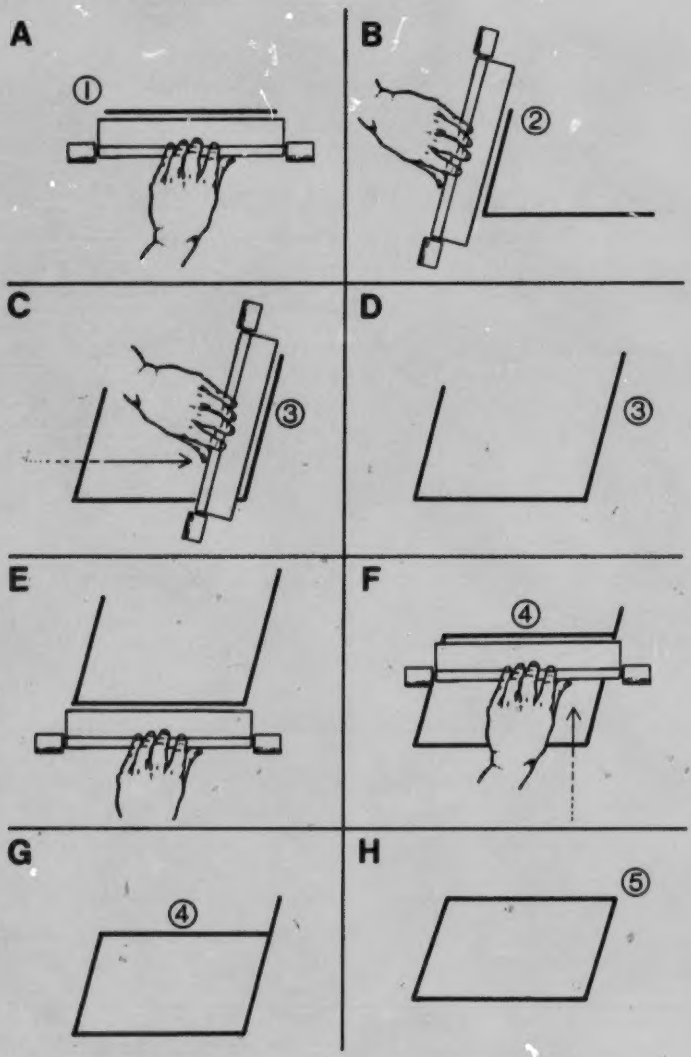
B) Then I draw another line segment ② starting at one of the endpoints.

C and D) Now I roll the translator to the other end of the first line segment and draw another line segment ③ .

E) I align the translator with the first line ① .

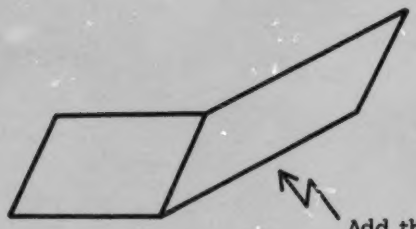
F and G) I roll the translator to the endpoint of the second line segment ② and draw the last segment ④.

H) I tidy the corners by erasing any pieces of lines which extend beyond the corners ⑤.



T: This shape is called a parallelogram because it has four sides and opposite sides are parallel. (Demonstrate again that the left side is parallel to the right side and that the top side is parallel to the bottom side.) Let's draw some more parallelograms.

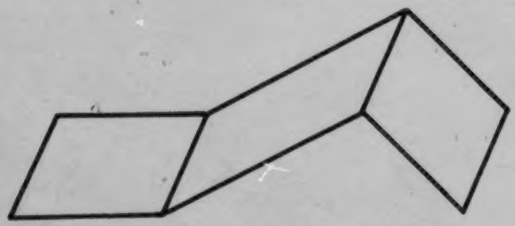
Draw another parallelogram attached to the first as in the next illustration. Again describe your technique throughout the construction.



Add this side first.

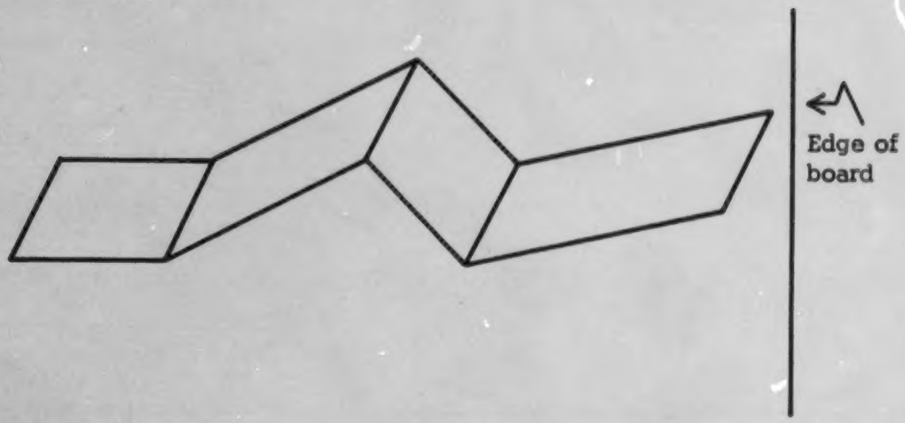
T: I want to draw another parallelogram and attach it to the one I just drew. Who can trace another parallelogram for us which is built on this side (point to the right side of your drawing)?

Have a student trace the outline of a parallelogram on the board. After someone has given a rough tracing, perform the construction yourself, again describing your moves. Be sure to emphasize aligning the translator, rolling, and then drawing a parallel line segment. Your drawing may look like this illustration.

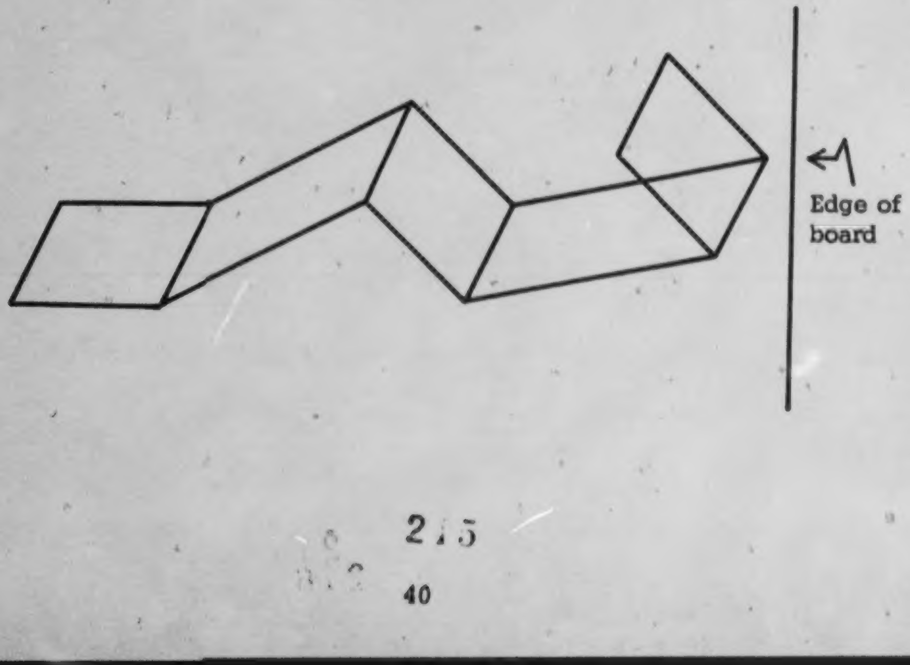


Continue to invite students to trace parallelograms for you to draw until the edge of the board is reached.

4 G



Again pose the problem of constructing a parallelogram on the right side, but still on the board. This problem is difficult for many students. Many will suggest a parallelogram be drawn in the small space between the end of the drawing and the edge of the board. Indicate that such an answer is correct, but that there is not enough room to maneuver the translator. Encourage discussion until a solution is found which reverses the direction of the construction as in the following illustration.

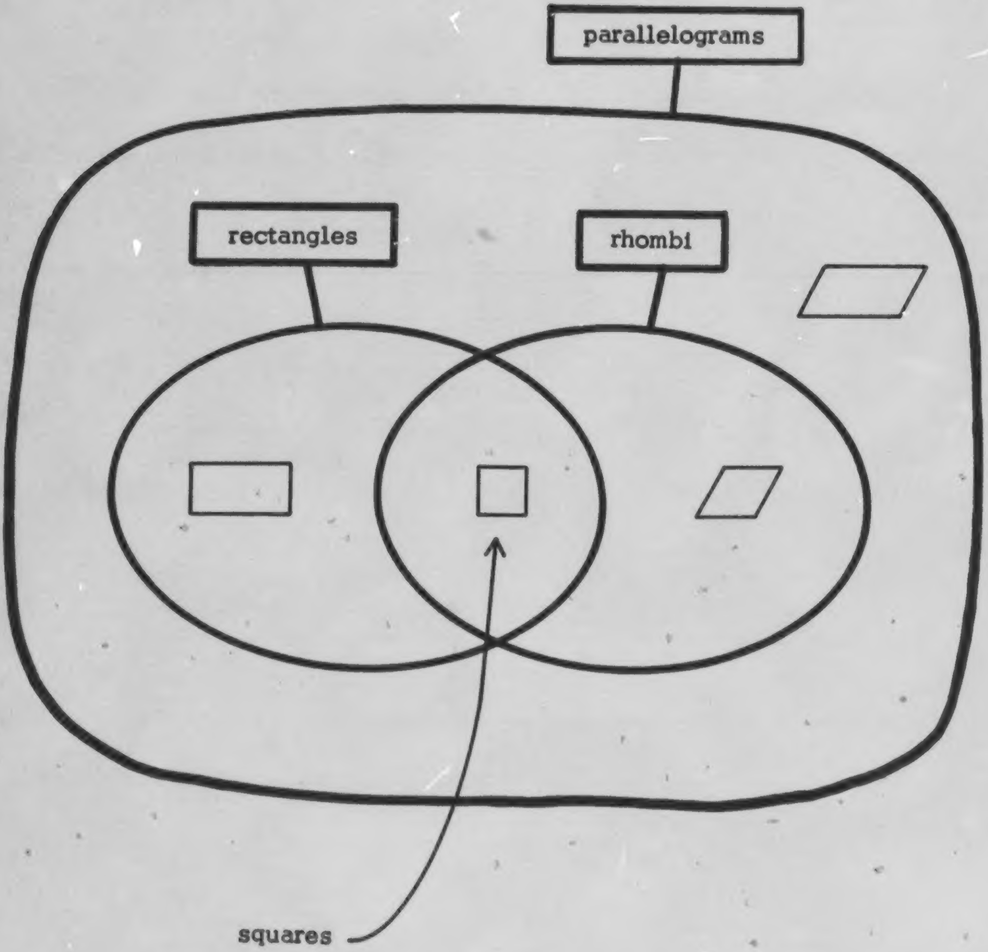


T: Look at the drawing. Who can find three line segments which are parallel to each other? Who can find three more?

Have students come to the board and point out these parallel line segments. Distribute translators and large sheets of unlined paper. Suggest to the students that they tape their paper to the desk with masking tape and provide them with some tape. Instruct the class to practice using the translator to draw sets of parallel line segments on the paper. Monitor the individual work and provide technical help as necessary. After individual practice on the paper, distribute Worksheets G17 and G18. Each worksheet is duplicated on the reverse to allow a second try after initial errors.

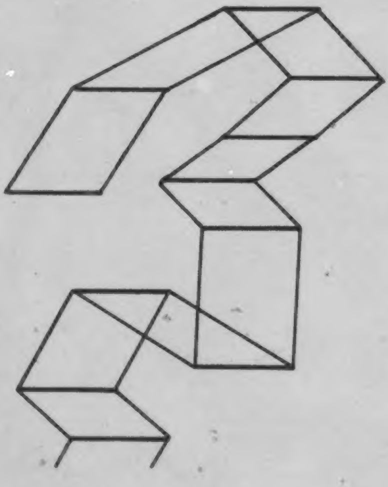
NOTE: The worksheet drawings include red line segments which do not appear in this lesson. They serve as a guide to constructing the parallelograms and also foreshadow the formal use of arrows in later lessons.

NOTE ON TERMINOLOGY: The parallelogram is the principal shape encountered in these lessons. A parallelogram is a four-sided figure with parallel opposing sides, but with no condition on the corner angles. There are two ways to specialize the definition: a parallelogram with four sides of equal length is called a rhombus, and a parallelogram with four angles of equal measure is called a rectangle. A parallelogram with four sides of equal length and four angles of equal measure is called a square. Thus, every square is a rectangle and every rectangle is a parallelogram. Also, every square is a rhombus and every rhombus is a parallelogram. In particular, every square is a parallelogram.


parallelograms
rectangles
rhombi
squares

Name________________ G 17 *

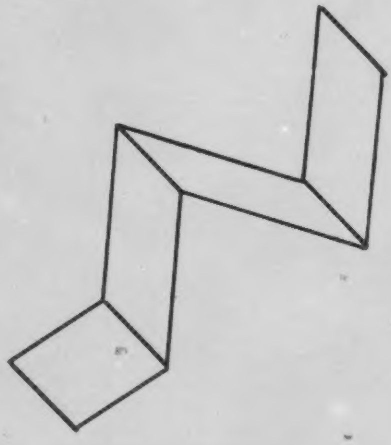
These parallelograms join the red line segments like the cars of a train. Use your translator to add at least six more red line segments and parallelograms to the train.



Many solutions are possible.

Name________________ G 18 **

Draw a train with at least four parallelograms connecting the two red line segments.



Many solutions are possible.

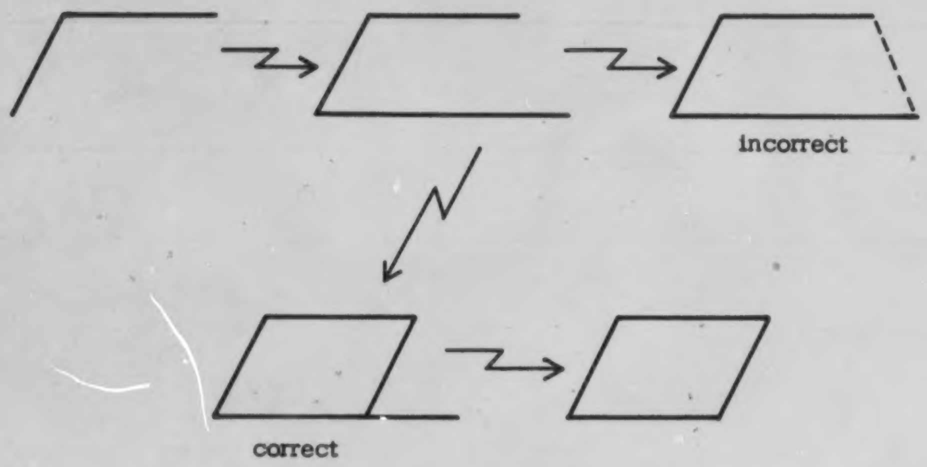
2.2 Parallelism #2

Review the discussion on usage of the translator in the introduction to this chapter before teaching the lesson.

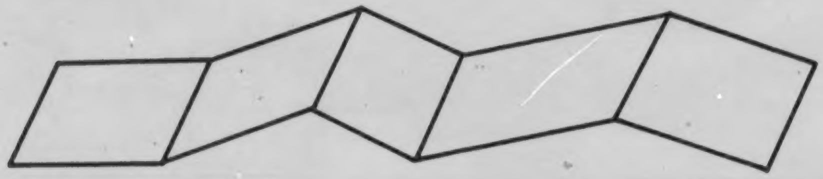
Begin by reviewing the basic parallelogram construction.

T: Help me to draw a parallelogram. I will draw two sides.

Draw two sides as illustrated. Invite a student to trace a third side.



Remind the class of using the translator to draw the third and fourth sides. When a student has given a rough tracing of a third side, use the translator to locate it precisely. Then invite another student to locate the fourth side. If you drew the third side too long or too short (which is to be expected), many students will simply join the endpoints of the segments as indicated by the dotted line in the incorrect completion (see illustration), rather than indicate the correct segment which is parallel to the opposite side. Request several tracings until a student is close to the correct answer. Then perform the construction with your translator, all the while describing your actions. Tidy the drawing by erasing the extra fragments. Continue to draw a train of parallelograms across the board.



T: Who can find two parallel line segments in the drawing?

Invite students to trace pairs of parallel line segments on the drawing.

T: Who can find a set of more than two line segments which are all parallel?

Continue the discussion until the full family of six (in our illustration) mutually* parallel line segments are found.

On another part of the board draw two dots.

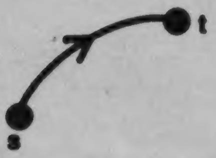
---

*A technical adverb denoting, in this case, that any two of the six line segments are parallel.

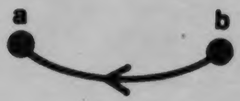
5G

T: Here are two points. Let's call one "s" and one "t". We can link two points together and call them a couple. To make a couple from two points, we need to pick a first point and a second point. We draw an arrow between the points to help us remember which is first and which is second.

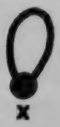
Draw an arrow from s to t. Then illustrate some other couples.



couple (s,t)



couple (b,a)



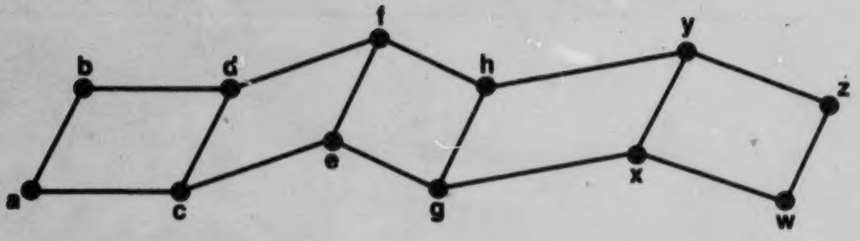
couple (x,x)

T: Another way of remembering is to write the names of the couples in a special way, using parentheses. If s is the first point and t is the second, we write "(s,t)" for the couple.

Repeat the explanation for "(b,a)" and "(x,x)". Return your class' attention to the parallelogram train on the board.

T: Let's give names to the points at the corners of the parallelogram train.

Name the points with letters or simple names suggested by the students. Emphasize each point by drawing a dot over its location.



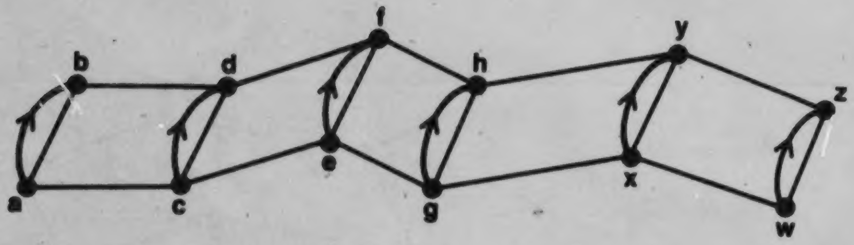
T: Who can name some couples in our drawing?

Affirm any correctly named couple. All that is required is for a student to name two of the points of the drawing. There are 144 couples in the illustration. However, the lines of the parallelogram train will tend to limit the students' perception of couples to those which are connected by one of the line segments. There are 32 such couples in our illustration.

When ten or fifteen couples have been noted, continue.

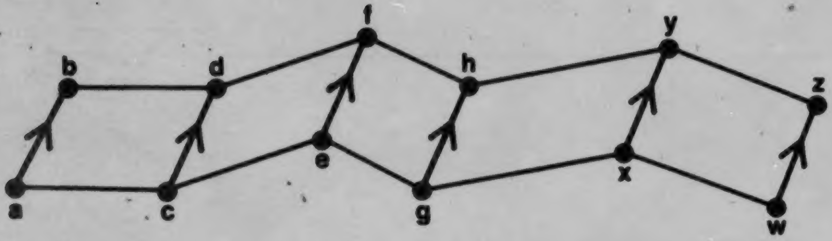
T: Let's draw red arrows for some of the couples.

Draw a red arrow for each of the couples in your drawing which corresponds to (a,b), (c,d), (e,f), (g,h), (x,y), (w,z) in the illustration and which already has been identified.



T: The couples marked with red arrows are related. The line segments which connect each of these couples are parallel and are connected by a parallelogram train and all of the arrows are pointing in the same direction. The starting point of an arrow (point to a) is connected to the starting point of the next arrow (point to c). The ending point of an arrow (point to b) is connected to the ending point of the next arrow (point to d). We say that such couples are equipollent. (Write this word on the board and have the students read it aloud several times.) All of the couples marked with red arrows are equipollent. Are there any other couples that should be marked with a red arrow? (If there are any more, mark them as they are identified.) We can make our drawing simpler by using the arrows as sides for the parallelograms.

Go back over the drawing and replace the red arrows already drawn with straight red arrows covering the parallel line segments as in the next illustration.



T: Now it is your turn to draw a parallelogram train and to find equipollent couples. I will put two problems on the board.

Distribute translators, large sheets of unlined paper, and colored pencils. Then write the following two problems on the board and read them to the class. Leave your own parallelogram train on the board for the students to consult while they work.

Problem 1

Draw a couple connected by a red arrow. Label the couple "(m,n)". Start at (m,n) and build a train with six parallelograms. Use red arrows to mark the couples equipollent to (m,n). Connect m with the starting point of the last couple in your train. Connect n with the ending point. What do you notice?

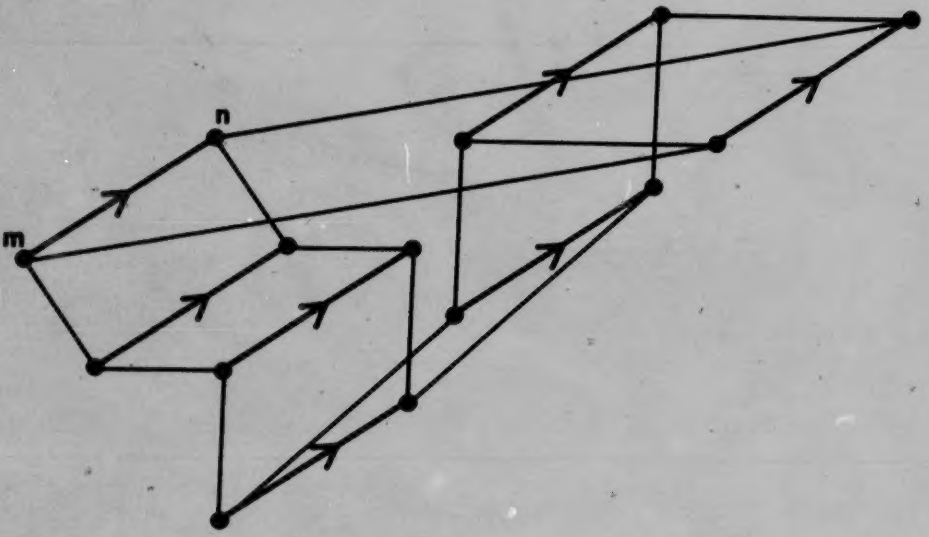
Problem 2

Draw two dots about 5 cm apart. Connect them with a blue arrow to indicate a couple. Name the couple "(x,y)". Find four couples equipollent to (x,y) and indicate them with blue arrows.

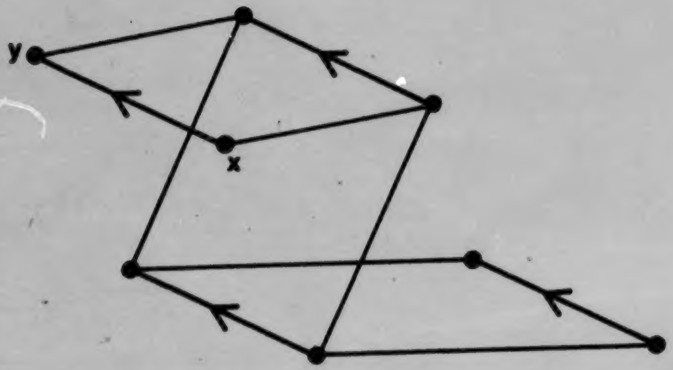
5G

Possible drawings for the problems are shown below.

Problem 1



Problem 2

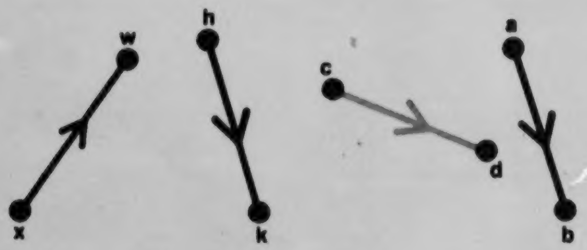


2.3 Parallelism #3

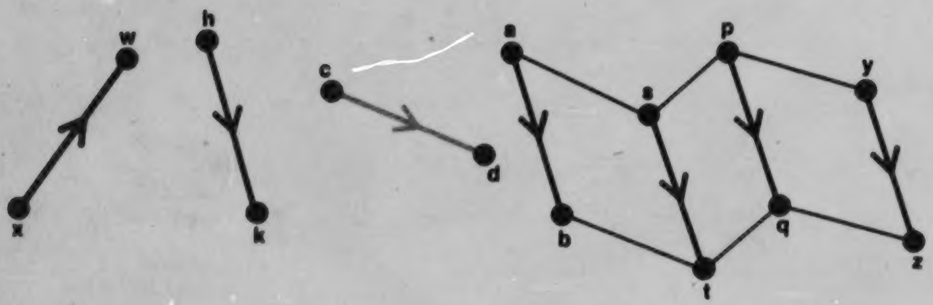
NOTE: In this lesson you will need two posters, Geometry Poster #1 and Geometry Poster #2. These are large easel pad size posters included in your classroom set of materials for CSMP Mathematics for the Intermediate Grades, Part II.

Display your translator and demonstrate its use by drawing several mutually parallel line segments. Then draw a line segment at an angle to the others and use the translator to show that it is not parallel to the others. All the while describe your movements and ask which segments are parallel to which other segments.

Now draw several pairs of points, one pair at a time. Name the two points and draw the line segment connecting them with colored chalk (a different color for each pair). Put an arrow head on the segment and ask the class to give the name of the couple thus formed.

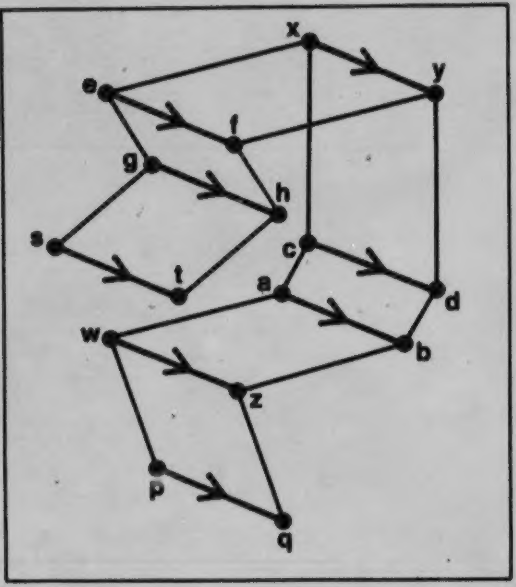


Emphasize the convention that the starting point of the arrow is the first component of the couple and the ending point of the arrow is the second component of the couple. Then choose one couple from which to start building a parallelogram train. Before you construct each parallelogram in the train, ask a student to first trace it on the board. And, as you perform the constructions, continually describe the action. The object is to construct three more couples equipollent to the starting couple. Your drawing might be similar to the following illustration.



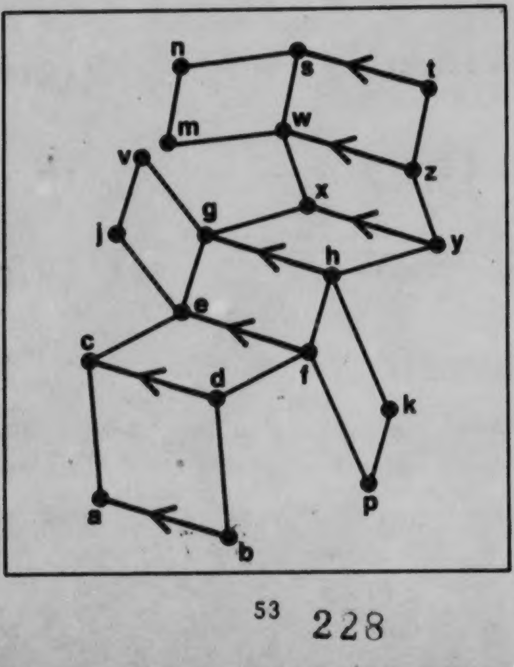
Emphasize that for equipollence to hold, the arrows must be pointing in the same direction; that is, all of the starting points of equipollent couples in the train must be linked (a to s, s to p, and p to y) and all of the ending points of equipollent couples in the train must be linked (b to t, t to q, and q to z). Ask whether (x,w) and (a,b) are equipollent, (h,k) and (x,w), (c,d) and (s,t), and so on. In each of these cases, demonstrate that the pairs of couples are not equipollent by using the translator to show that their lines are not parallel. Ask whether (t,s) is equipollent to (a,b). Emphasize that they are not equipollent because the starting point of (t,s) is linked to the ending point of (a,b); that is, t is linked to b.

Display Geometry Poster #1.


x
y
e
f
g
h
s
t
c
a
d
w
b
z
p
q

Ask students to name the couples which are identified by arrows in the drawing; then to count the number of parallelograms linking various pairs of couples (e.g., (e,f) and (w,z) are linked by a train of four parallelograms); then to identify pairs of equipollent couples.

Display Geometry Poster #2.


n
s
t
m
w
v
z
x
g
j
y
h
e
c
d
f
k
p
a
b

6G

Ask the class to name the couples which are identified by arrows. There are seven given by red arrows, namely,

(b,a); (d,c); (f,e); (h,g); (y,x); (z,w); (t,s).

T: Are these couples equipollent? (Yes)

Could we indicate other couples in this drawing?

S: Yes, we could add more arrows to show more couples.

Accept all suggestions for couples if they are correct, but only draw arrows for the equipollent couples (n,m), (s,w), (t,z) and for the equipollent couples (v,j), (g,e), (h,f), (k,p). This will delineate two more equipollent families. Use a different color for each family. Color the first member of each family that is mentioned. When another is suggested, ask what color to use and which direction to point the arrow. Make lists of equipollent couples on the board as you progress. Make a dot of the appropriate color next to the name of the couple.

● **(n,m)**
● **(s,w)**
● **(t,z)**

● **(v,j)**
● **(g,e)**
● **(h,f)**
● **(k,p)**

Distribute translators, large sheets of unlined paper, and colored pencils, and instruct the students to work on these two problems.

Problem 1

Draw two points 7 cm apart. Name them. Draw a red arrow to indicate a couple. Find eight other couples equipollent to your couple. Indicate them with red arrows.

Problem 2

On the same paper, draw a blue couple not equipollent to your red couple. Find five other couples equipollent to the blue couple and indicate them with blue arrows.

Circulate and assist students who have difficulty using the translator. Encourage them to make accurate drawings.

NOTE: Save Geometry Poster #2 for use in Section 2.4.

2.4 Parallelism #4

Draw Figure 1 (see page 57) on the board before the lessons begins. Note that the distance from x to y must be the same as the distance from a to b.

Begin the lesson by reviewing notions of parallelogram, couple and equipollence.

T: What is a couple?

S: Two points.

Draw two dots on the board.

T: Is this a couple?

S: No, you need an arrow to show which is the starting point and which is the ending point.

Draw an arrow.

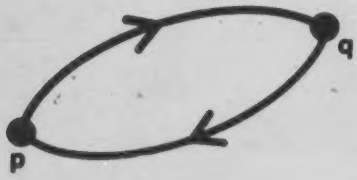
T: What is a name for this couple?

S: Call the starting point "p" and the ending point "q", then the couple is called "(p,q)".

T: Can I indicate another couple with these dots?

S: Yes, (q,p). Draw an arrow from q to p.

Draw an arrow in a different color for the opposite couple.



Display Geometry Poster #2. Ask students to name equipollent couples. See the discussion of Geometry Poster #2 in Section 2.3 for several families of equipollent couples which are shown on this poster. Then take down the poster and refer to your drawing of Figure 1.

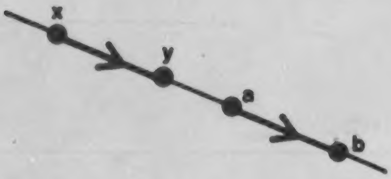


Figure 1

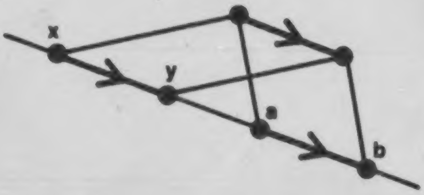


Figure 2

T: Here I have two couples (x,y) and (a,b). Why do you think that I have indicated them with the same color arrow?

S: They are equipollent. Equipollent couples are always indicated by the same color arrow.

T: How can we show that they are equipollent?

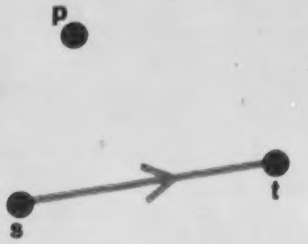
S: We need to build a parallelogram train from one couple to the other.

T: Who can trace a parallelogram train from (x,y) to (a,b)?

Encourage discussion of the problem and let the class attempt to find a solution by trial and error. If no one provides a solution now, leave the problem unsolved for further discussion. For your information, one possible solution is shown in Figure 2.

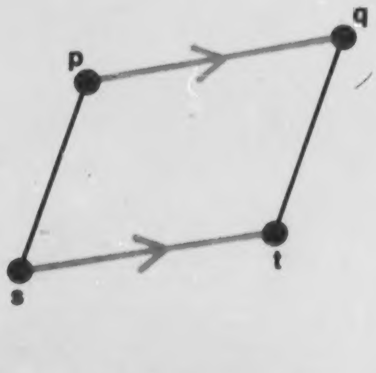
Do not erase your drawing of Figure 1.

On the board, draw a point p and a couple (s,t) not equipollent to those in Figure 1. Use a different color than that used in Figure 1.

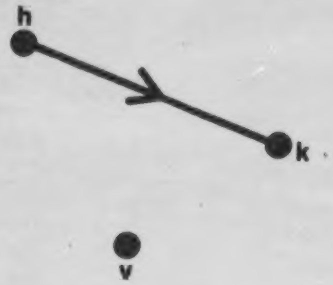


T: I want to find a couple equipollent to (s,t) with p as the starting point. Who can help me?

Ask a student to trace the required parallelogram and then perform the construction as illustrated below. Emphasize that the first step of the construction is to connect the starting point s to the starting point p.

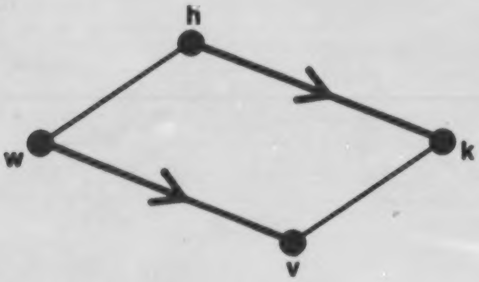


Without disturbing the other drawings, draw another couple (h,k) and point v.



T: I want to find a couple equipollent to (h,k) with v as the ending point. Who can help me?

Again ask a student to trace the required parallelogram and then perform the construction as illustrated below. Emphasize that the first step of the construction is to connect the ending point k to the ending point v.



Display Geometry Posters #3 and #4. Distribute translators, large sheets of unlined paper, and colored pencils. Read the directions for all four problems to the class, and then instruct the students to replicate and solve the problems on their papers.

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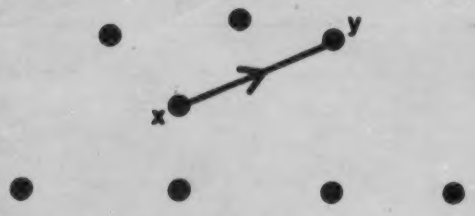
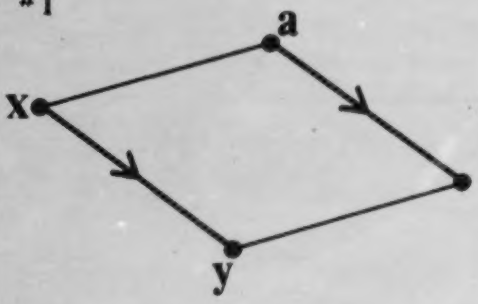
If some students have time to work on another problem, draw this picture on the board and write the problem next to it.



Use these points to draw red couples equipollent to (x,y). Use some of the points as starting points and some as ending points. Try to use some points as both ending and starting points.

Solutions for Geometry Poster #3

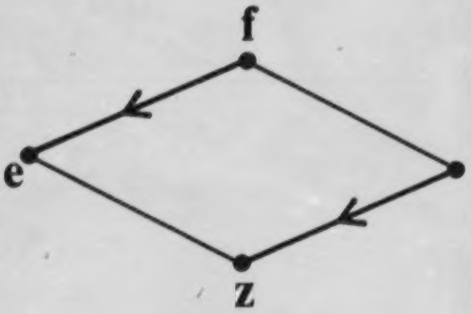
#1



(x,y) is equipollent to (a,?)

Find the missing point.

#2



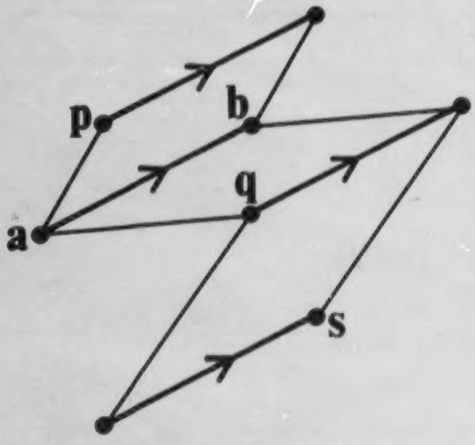
(f,e) is equipollent to (?,z)

Find the missing point.

Students may label the points they find with any names they wish.

Solutions for Geometry Poster #4

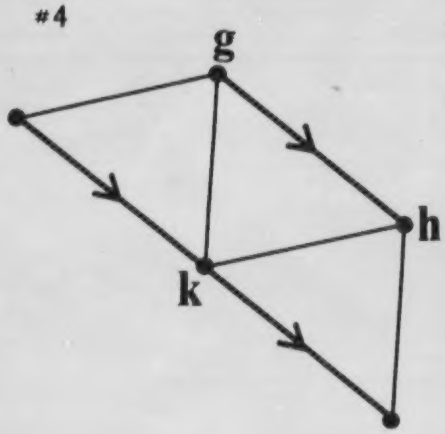
#3



(a,b) is equipollent to (p,?), (q,?) and (?,s)

Find the missing points.

#4



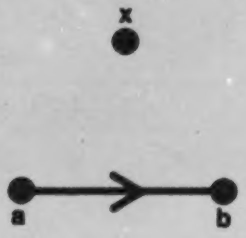
(g,h) is equipollent to (k,?) and (?,k)

Find the missing points.

Students may label the points they find with any names they wish.

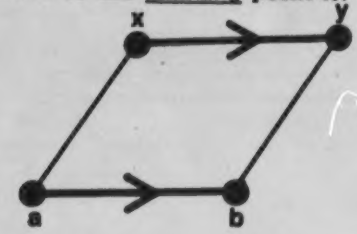
2.5 Parallelism #5

Draw a couple (a,b) on the board and a third point x as shown.



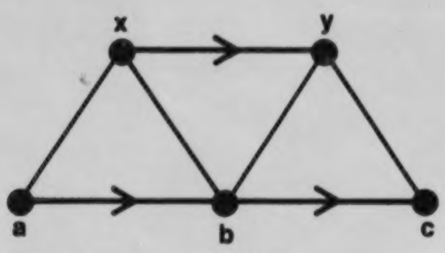
T: I want to find a couple equipollent to (a,b) that starts at x.

Invite a student to trace the parallelogram and then perform the construction as shown below. Emphasize that the first step in the construction is to connect the starting point a to the starting point x.



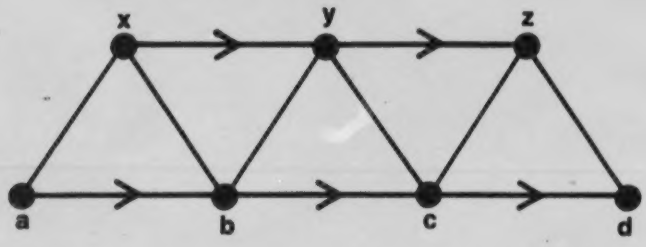
T: Now I want to find a couple equipollent to (a,b) that starts at b.

This construction is difficult for many students. Encourage discussion and suggestions from the class. If necessary, suggest that they construct a parallelogram train to complete the couple, and that (x,y) can be used since it is equipollent to (a,b). Remind them also that the starting points of equipollent couples are connected in the parallelogram trains, so connecting x and b would be a good beginning for the construction. At each stage of this problem, reiterate this suggestion of beginning the construction by connecting starting points. Continue until someone suggests a correct construction as shown in the next illustration.



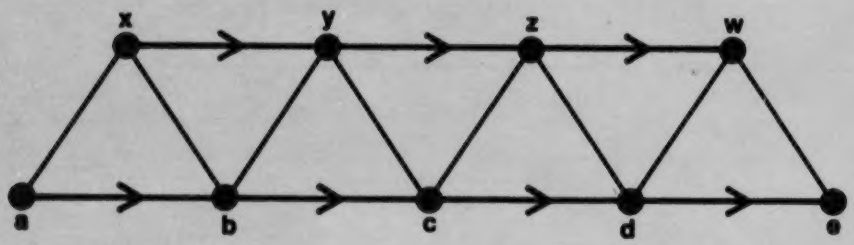
T: Now I want to find a couple equipollent to (a,b) that starts at c.

Encourage discussion of a two step construction, first constructing a couple starting at y and then one starting at c. Ask students to trace the parallelograms involved before you perform the construction.

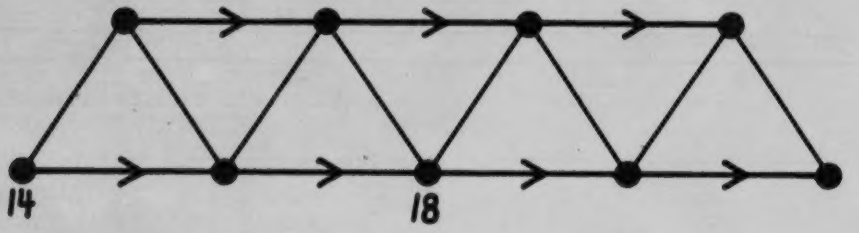


T: Let's continue the construction one more step and find a couple equipollent to (a,b) that starts at d.

Again encourage discussion until a student suggests a two step construction shown below and then complete it.



Ask students if the picture reminds them of anything . . . a number line? Erase the letter labels on the points and replace them with numerals as shown below.



T:	What number is here (point to the dot between 14 and 18)?
S:	16.

Repeat several times this exercise of finding the number in the middle. Each time choose a pair of numbers, say 37 and 41, to replace 14 and 18, respectively. Ask students to calculate mentally the number in the middle. You can adjust your choices of pairs of numbers to the numerical ability of your class.

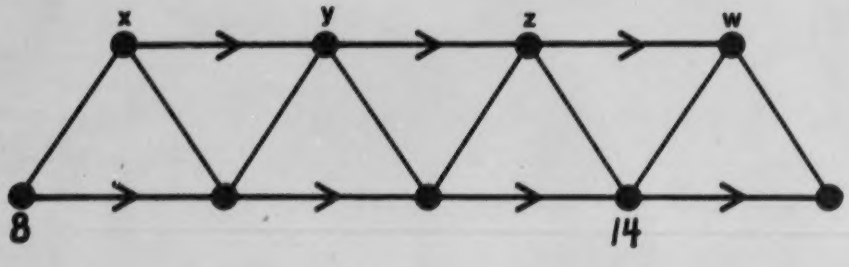
After five or six of these problems, relabel the points "a", "b", etc. as before.

T: We say that b is the midpoint of a and c because (a,b) and (b,c) are equipollent. What is the midpoint of a and c? (Answer: b)

What is the midpoint of x and z? (Answer: y)

What is the midpoint of b and d? (Answer: c)

Erase the labels "a", "b", "c", "d" and "e", replacing "a" with "8" and "d" with "14".



Ask the class to find the numbers corresponding to the other dots. (Answer: b = 10, c = 12, e = 16)

Repeat this exercise with 2 and 3.5. (Answer: b = 2.5, c = 3, e = 4)

Repeat the exercise a few more times. Choose numbers suited to the abilities of your class. Label a different pair of dots each time. Always place both numbers on the same line with the smaller to the left of the larger.

10G

Erase the numerals on your number line and relabel the first two points "a" and "b" as before.

Distribute translators, large sheets of unlined paper, and colored pencils.

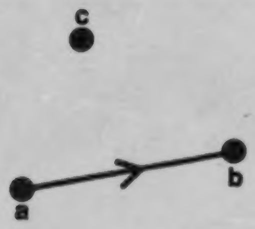
T: Now it is your turn to draw a number line the same way as I have drawn mine. Draw one couple on your paper to start, and then build a parallelogram train like the one on the board. Remember: connect starting point to starting point and ending point to ending point. When you are finished, put a number for a and a larger number for b. Then find the correct number for the other dots on your number line.

Encourage students to go on with the construction - to construct "longer" number lines.

2.6 Parallelism #6

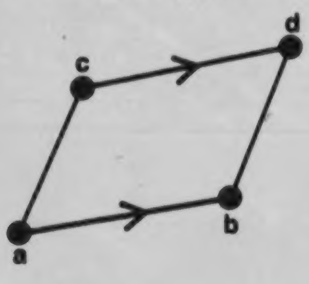
Begin the lesson by reviewing the constructions in Section 2.4.

Draw a couple (a,b) on the board and a third point c as below.

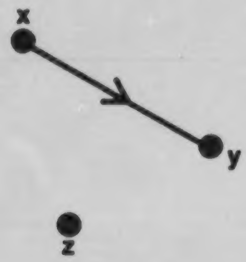


T: I want to find a couple starting at c that is equipollent to (a,b).

Ask a student to trace the required parallelogram and then perform the construction. Remind the student to first connect the starting point a to the starting point c.



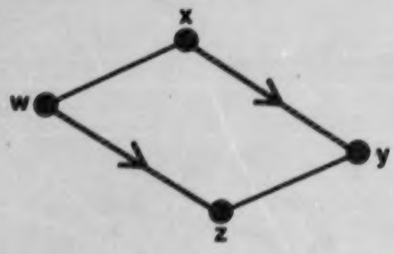
Continually describe your actions. Draw another couple (x,y) and another point z as below.



T: I want to find a couple ending at z that is equipollent to (x,y).

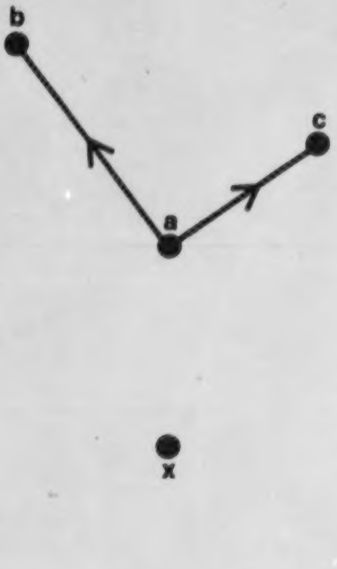
11G

Again ask a student to trace the required parallelogram and then perform the construction. Remind the student to first connect ending point y to ending point z.

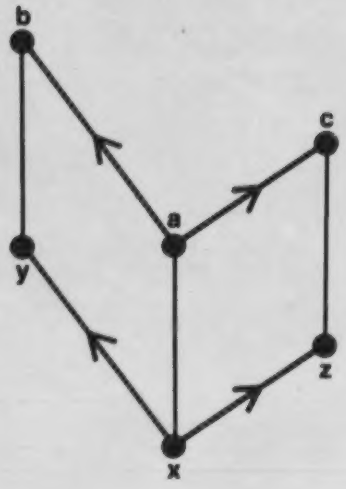


The remainder of the period is to be devoted to a more complicated construction.

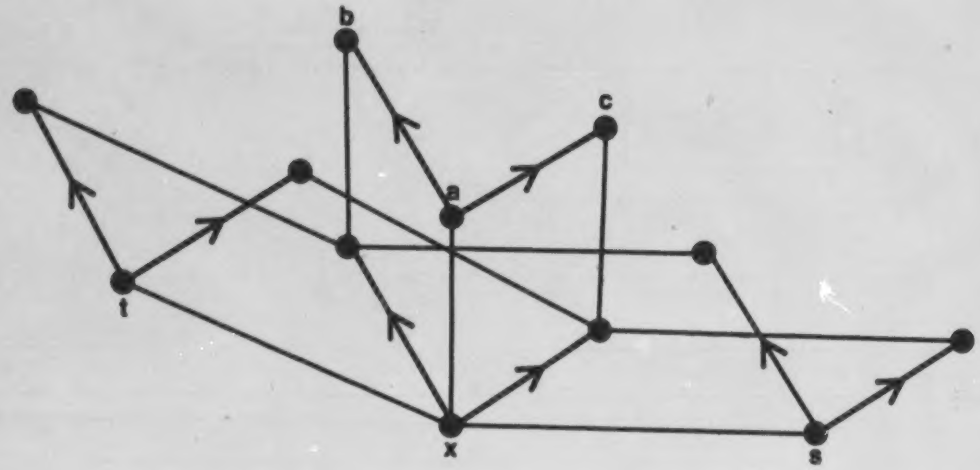
Distribute translators, large sheets of unlined paper, and colored pencils. Draw this picture on the board and have students copy it carefully on their papers.



Ask the students to construct two couples starting at x, one that is equipollent to (a,b) and one that is equipollent to (a,c). A proper construction is shown below.



This is a challenging construction for most students because of the complexity of the double parallelogram construction. For those who finish, add another point t to the left of x and ask for two couples starting at t, one that is equipollent to (a,b) and one that is equipollent to (a,c). If anyone finishes the second construction, add yet another point to s to the right of x and ask for two couples starting at s, one that is equipollent to (a,b) and one that is equipollent to (a,c). One possible construction for these two extra points is illustrated below.



At the end of the lesson, work out this construction at the board with the assistance of some students.

2.7 Parallelism #7

Draw two points on the board. Label them "a" and "b".

T: What do we have here? (Two points)

What can we make with them? (A couple)

How many couples? (Two)

Draw an arrow from a to b.

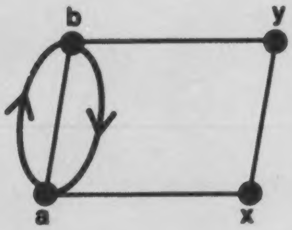
T: What does the arrow tell us? (Which dot is first)

What is the name of this couple? ("(a,b)")

Draw an arrow from b to a.

T: What does this arrow show us? (The other couple, (b,a))

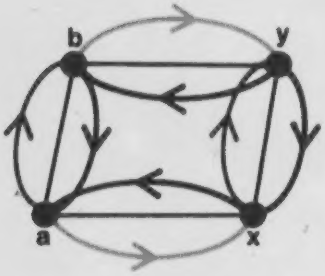
Construct a parallelogram and label the corner points. Be sure to describe the action once again.



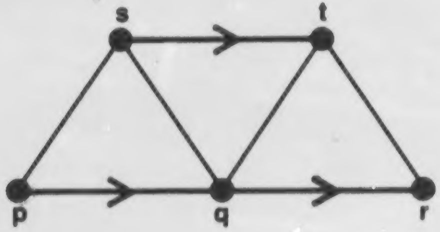
Ask the students to name couples that can be formed with the corners of the parallelogram.

After five or six have been named, ask if there are any pairs of equipollent couples in the collection. As they are identified, draw arrows in the picture.

A complete naming of pairs of equipollent couples would yield this picture. Use a different color for each pair of equipollent couples.



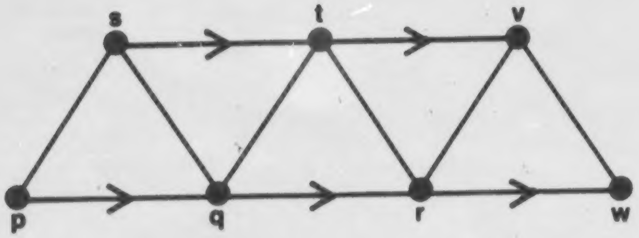
Now construct the figure below, starting with the segment connecting p and q, as developed in Section 2.5 of this chapter.



Discuss your construction as you perform it and ask students to trace the parallelograms before you draw them. Do several midpoint problems such as the following. (Student responses are boxed.)

p	q	r
5	$\boxed{15}$	25
36	$\boxed{46}$	56
64	$\boxed{84}$	104
1.5	$\boxed{2.5}$	3.5

Continue the parallelogram train one more step.



12G

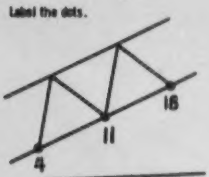
Do several number line problems such as the following. (Student responses are boxed.)

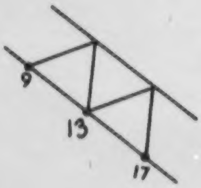
p	q	r	w
8	$\boxed{11}$	$\boxed{14}$	17
26	$\boxed{40}$	$\boxed{54}$	68
3.7	$\boxed{4.5}$	$\boxed{5.3}$	6.1
1.36	$\boxed{1.65}$	$\boxed{1.94}$	2.23

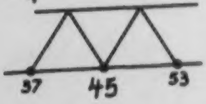
Worksheets G19, G20, G21, and G22 are available for the rest of the lesson.

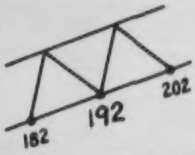
Name ____________________ **G 19** *

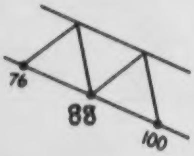
Label the dots.

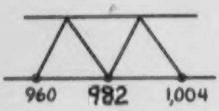






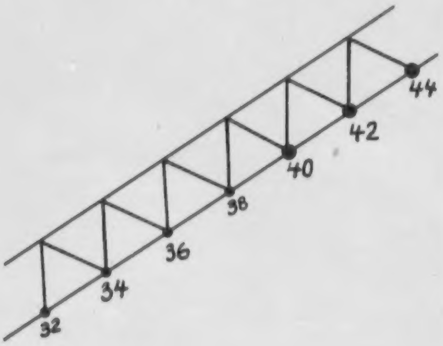





960
982
1,004

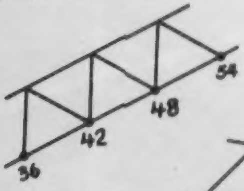
Name ____________________ **G 20** **

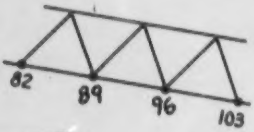
Label the dots. Use your translator to place a dot correctly for 44.

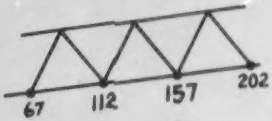

32
34
36
38
40
42
44

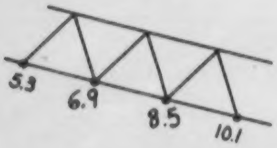
Name ____________________ **G 21** ***

Label the dots.



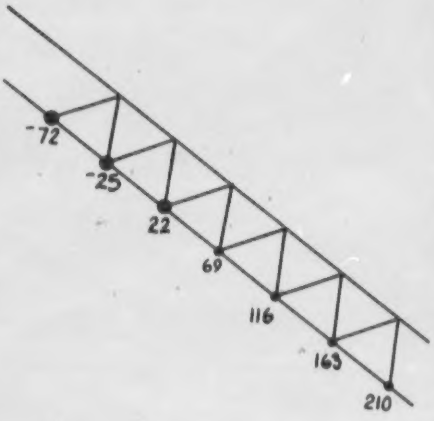



67
112
157
202



Name ____________________ **G 22** ****

Label the dots. Use your translator to place correctly a dot for ⁻72.


⁻72
⁻25
22
69
116
163
210

PROBABILITY AND STATISTICS

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades
Part II

Probability and Statistics

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR P – LESSONS

WEEK	Probability and Statistics Geometry and Measurement	Materials
1	ADJUSTMENT DAY	
2	1G G1.1: Length #1 p. 5	Teacher: 30 cm ruler Student: 30 cm ruler; colored pencils; Worksheets G01, G02, G03
3	2G G1.2: Area #1 p. 10	T: Grid board; colored chalk S: Colored pencils; Worksheets G04, G05, G06
4	3G G1.3: Area #2 p. 14	T: Grid board; colored chalk S: Colored pencils; Worksheets G07, G08, G06
5	1P P1.1: Making a Fair Choice p. 3	T: Marbles (two different colors); code names (optional); gift (optional) S: None
6	2P P1.2: Probabilities p. 21	T: Marbles (two different colors); colored chalk S: Colored pencils; Worksheets P01, P02
7	4G G2.1: Parallelism #1 p. 35	T: Demonstration translator; colored chalk S: Translator; large sheets of unlined paper; colored pencils; Worksheets G17, G18
8	5G G2.2: Parallelism #2 p. 44	T: Demonstration translator; colored chalk S: Translator; large sheets of unlined paper; colored pencils
9	6G G2.3: Parallelism #3 p. 51	T: Demonstration translator; colored chalk; IG-II Geometry Posters #1, #2 S: Translator; colored pencils; large sheets of unlined paper

SUGGESTED SCHEDULE FOR P–LESSONS

WEEK	Probability and Statistics Languages of Strings and Arrows	Materials
10	ADJUSTMENT DAY	
11	**3P** P1.3: Fair Game? #1 p. 29	T: Colored chalk S: None
12	**4P** P1.4: Fair Game? #2 p. 39	T: Dice (two different colors); Colored chalk S: Worksheets P03, P04
13	**9L** L1.4: Arrow Messages #1 p. 46	T: Colored chalk S: Colored pencils; paper
14	**10L** L2.6: Logical Thinking #2 p. 107	T: The String Game; colored chalk S: None
15	**5P** P1.5: Fair Game? #3* p. 50	T: Colored chalk; Teacher's Guide pages 66, 67, and 68. S: None
16	**12L** L1.5: Arrow Messages #2 p. 61	T: Colored chalk S: Colored pencils; Worksheets L11, L12, L13, L14, L15, L16
17	**13L** L2.8: String Game Analysis #5 p. 119	T: The String Game S: String Game Analysis Pad Worksheets L21, L22
18	ADJUSTMENT DAY	

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* In the Master Suggested Schedule this lesson appears under the heading "Geometry and Measurement, Probability and Statistics". To avoid printing an extra schedule page, we include it here so that you are aware that there is a P-lesson scheduled in the 15th week.

SUGGESTED CHRONOLOGICAL ORDER OF P—LESSONS

Lesson One	P1.1 - Making a Fair Choice	3
Lesson Two	P1.2 - Probabilities	21
Lesson Three	P1.3 - Fair Game? #1	29
Lesson Four	P1.4 - Fair Game? #2	39
Lesson Five	P1.5 - Fair Game? #3	50

PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise when studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside itself—to other disciplines, as well as more generally to understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful application can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

The lessons of this strand deal with systematic methods of analyzing games of chance and of finding numerical probabilities for outcomes or events. These situations extend types of activities similar to those presented in Part I of the CSMP Intermediate Grades Program.

1

GAMES OF CHANCE

This series of lessons deals with problems or games related to random experiments. In the process of analyzing these experiments, the students are introduced to and use cord diagrams, probability trees, and other pictorial methods. They begin in a natural way to think about order among fractions and to multiply fractions. They meet informally (not explicitly) for the first time an important method in probability, the multiplication principle for finding probabilities in a multi-stage experiment.

The first lesson concerns situations in which a random device with two equally likely outcomes is used to make a choice between several children in such a way that all the children have the same chance of being chosen; that is, so that there is a fair choice. The random device used is a pair of marbles, one black and one white, but it could as well be a coin or a symmetric 0, 1-spinner.

The second lesson reviews the methods from Chapter 1 of the Probability and Statistics strand in Part I of the CSMP Intermediate Grades Program. In this lesson probabilities are used to describe one outcome as being more or less favorable than another.

The third and fourth lessons consist of a series of games of chance described in a continuing story about three brothers. One of the brothers likes to invent these games and to "trick" his brothers into playing games which favor the inventor. The other two brothers, being always suspicious, subject the games to analysis and compare their respective probabilities of winning.

The first of these (Lesson 3) is played by spinning two different 0, 1-spinners (a two-stage experiment) and adding the results. One of the outcomes (sums) 0, 1 and 2 is winning for each of the three brothers. The game is analyzed and found to have different probabilities for the outcomes (not a fair game). Another game is a two-stage experiment involving varying numbers of white and black marbles. It, too, is analyzed for fairness.

Analysis of a game played with two dice (Lesson 4) leads to a representation of the outcomes by means of a coordinate grid.

The last lesson concerns some two-stage experiments with marbles, the analysis of which uses still another kind of picture. The method of analysis suggested in this lesson presents an alternative to the probability tree method used in previous lessons.

CAPSULE LESSON SUMMARIES

Section 1.1: From a pair of marbles, one black and the other white, choose one at random. Use this random device to choose one child from a given number of children so that they all have the same probability of being chosen; that is, so that the choice is fair.

Section 1.2: From a set of two white and two black marbles, select two at random and find the probabilities of 0 blacks being drawn, of 1 black being drawn, and of 2 blacks being drawn. Then change the experiment by changing the number of black and white marbles.
(Worksheets P01, P02)

Section 1.3: Spin two different 0, 1-spinners and find the probabilities of the three outcomes: the sum of the numbers obtained is (i) 0, (ii) 1, (iii) 2. Analyze this experiment with a probability tree.

Section 1.4: Roll two dice, one black and one red, and represent the outcomes as points on a coordinate grid. Then find the probabilities of various sums of the numbers appearing.
(Worksheets P03, P04)

Section 1.5: Draw one marble from a first bag containing 1 white and 2 red marbles; put that marble into a second bag containing 1 red and 2 white marbles; then draw one marble from the second bag. Find the probability that the second marble drawn is white and the probability that it is red. Vary the experiment by changing the initial number of marbles in the bags. Try to find a distribution of marbles which equalizes the probabilities.

1.1 Making a Fair Choice

For this lesson, you will need one black and one white marble. If these colors are not available, use any two different colors and change the colors in the lesson accordingly.

Exercise 1

Make the indicated motions as you say:

T: I have two marbles, one black and one white. I hold them with both hands behind my back and shake them; then I put one in my right hand. What is the probability that a black marble will be in my right hand? (Hold out your closed right hand.)

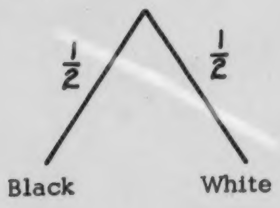
Accept any reasonable response.

S: 1 out of 2 (one-half; 1 chance in 2; 1 in 2; etc.)

T: What is the probability that it will be white?

S: The same; 1 out of 2.

T: Here is a way of recording the probabilities (draw the picture on the board). It is called a "probability tree".



T: We are going to use these marbles to help us make choices. But, before we do let's look at some other devices used to make choices.

Do you know how a referee at a football game decides which team will kick off?

S: He flips a coin.

T: Can you explain how this helps him decide?

S: One of the teams chooses heads. When the referee flips the coin, if it comes up heads, then that team can decide to kick or receive.

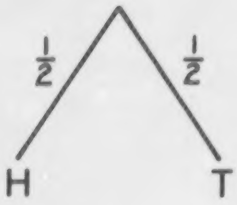
T: Is that fair?

S: Yes; both teams have the same chance of winning the toss.

T: What is the probability the coin will come up heads? (1 out of 2 or $\frac{1}{2}$)

What is the probability it will come up tails? (1 out of 2 or $\frac{1}{2}$)

Draw a probability tree on the board.



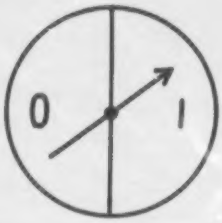
T: Does each team have the same probability of winning the toss? (Yes)

So, it is fair.

Could the referee use the two marbles to make the decision? (Yes; one team chooses white and the other black.)

Would the choice still be fair? (Yes; both colors have the same probability.)

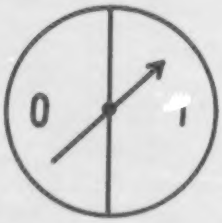
Draw this picture of a 0, 1 - spinner on the board.

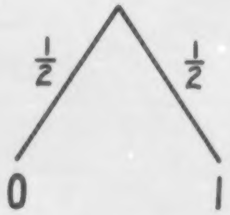


T: Imagine that you spin this spinner. What is the probability the pointer will stop on "1"? (1 out of 2 or $\frac{1}{2}$)

On "0"? (1 out of 2 or $\frac{1}{2}$)

Draw a probability tree next to the spinner.





1 P

T: Could the football referee also use this 0,1-spinner? (Yes; one team chooses 0 and the other 1.)
Would the choice still be fair? (Yes; both teams would have the same chance of winning the spin.)

T: Now, suppose I spin this spinner 100 times. What do you think will happen?

S: Close to 50/50.

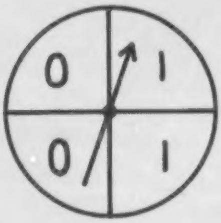
T: What does that mean?

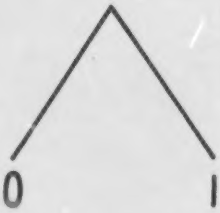
S: About 50 times you will get 0 and about 50 times you'll get 1.

T: Suppose I flip a coin 200 times. What do you think will happen?

S: You'll get about 100 heads and 100 tails.

Draw this picture of a 0,1-spinner and tree on the board.





T: Imagine you spin this spinner. What is the probability that the outcome will be 0; that is, the pointer stops on "0"?

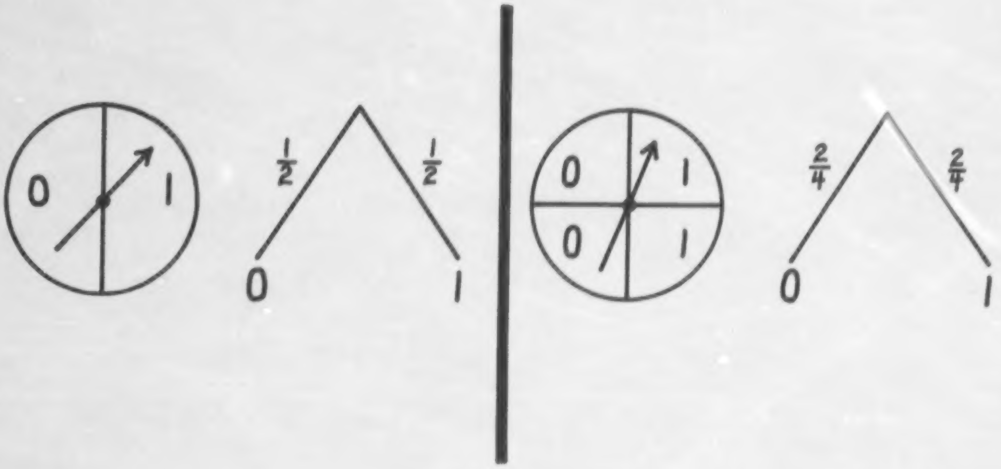
S: 2 out of 4 (or $\frac{2}{4}$).

T: What is the probability the outcome will be 1?

S: The same; 2 out of 4 (or $\frac{2}{4}$).

Have a student write these probabilities on the tree.

You should have two spinner pictures and two probability trees on the board.



T: Suppose you are playing a game in which you win when the outcome is 1. Which of these spinners would you prefer to use?

S: The second; there are two chances to get a 1 and only one chance with the first spinner.

S: Yes, but with the second spinner it is 2 out of 4 and with the first spinner it is 1 out of 2.

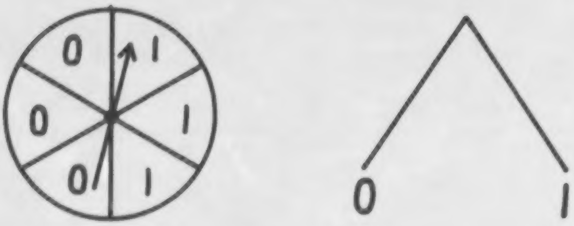
S: They are the same; 2 chances out of 4 is the same as 1 chance out of 2.

1P

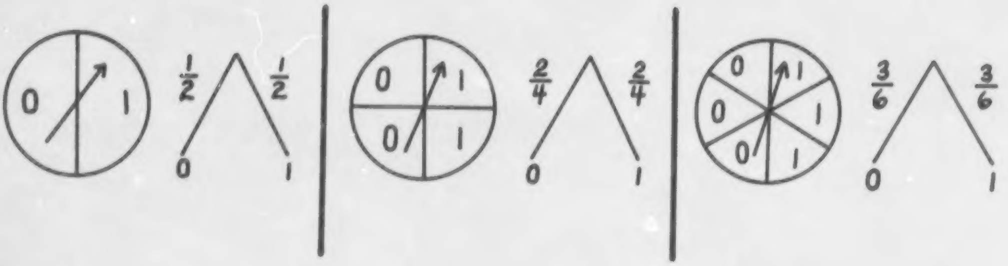
Write this information on the board.

$$\frac{2}{4} = \frac{1}{2}$$

Draw another 0, 1 - spinner and tree,



and repeat the questions for this situation. The discussion should lead to this information.


0
1
$\frac{1}{2}$
$\frac{1}{2}$
0
1
0
1
0
1
$\frac{2}{4}$
$\frac{2}{4}$
0
1
0
1
0
1
0
1
$\frac{3}{6}$
$\frac{3}{6}$
0
1

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$$

Invite the students to draw or imagine other 0,1-spinners giving equal probabilities for 0 and 1. For each new spinner, draw the corresponding probability tree and add the information to your string of equalities. After a few minutes, you should have a rather lengthy statement of equalities. For example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{6}{12} = \frac{10}{20} = \frac{50}{100} = \frac{500}{1{,}000} = \frac{1{,}000{,}000}{2{,}000{,}000}$$

Exercise 2

Tell a story about a father who has two children. He arrives home from a trip and finds that he has lost one of the two presents he bought. How can he give the remaining gift (it cannot be broken in half) to one of his two children in a fair way? You could dramatize the situation by choosing two students to be the children (say, "A" for "Amy" and "B" for "Brian").

T: How can I decide to which of you I will give the gift? It has to be fair; each of you should have the same chance of getting it.

The class most likely will suggest tossing a coin, or using one of the 0,1-spinners, or using the two marbles.

T: We will use the two marbles. Amy, which color marble do you want? (White) Then Brian gets black. If the marble in my right hand is white, Amy gets the gift and if it is black, Brian gets the gift. Is this fair?

S: Yes, they both have the same chance to get the gift.

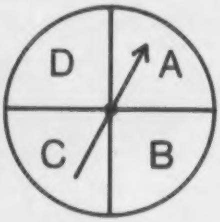
If the students do not mention "probability", you should use this word in a statement such as

T: It is fair because they both have the same probability of getting the gift.

The students will pick up the word if you continue to use it in passing.

T: Suppose now that the father has four children, A, B, C and D for short, and still only one gift. How can he give it to one of them in a fair way?

A student might suggest using a spinner like this.

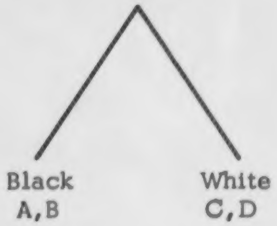


T: That would be a good idea, but suppose the father has only a black and a white marble. How can he use the marbles?

It is likely that after a little discussion of the situation, a student will suggest first dividing the children into two groups, say A and B together and C and D together. Use the marbles to choose one of the two groups. Then use the marbles to pick one of the children from the chosen group. If you do not get a similar decision method, ask some questions which direct the class to think about this method.

You could again dramatize the situation letting four students play the four children. Have them divide into two groups. Let one group choose its color.

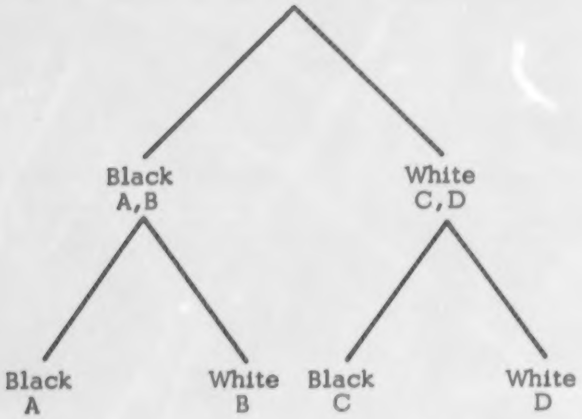
Draw this tree on the board (say, A and B have chosen black).


Black
A,B
White
C,D

T: If the outcome is black, then A,B is chosen. If the outcome is white, then C,D is chosen. Then, what do we do?

S: Use the marbles again to choose between A and B, or to choose between C and D.

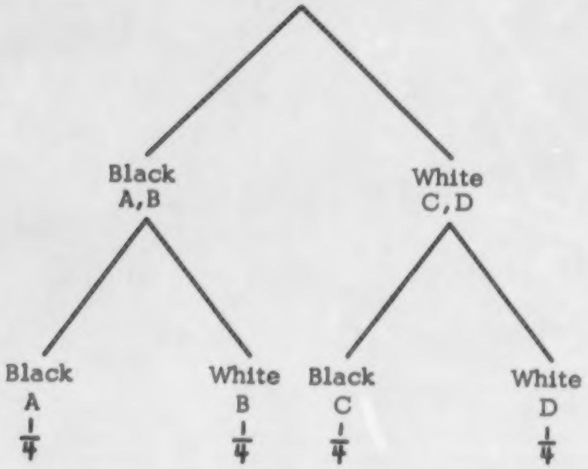
Draw the second stage of the tree, using student suggestions.


Black
A,B
White
C,D
Black
A
White
B
Black
C
White
D

T: Is this a fair method? Do all the children have the same probability of getting the gift?

S: Yes, they all have 1 chance in 4.

Write "$\frac{1}{4}$" at the bottom of each branch of the tree.



Ask the class to watch you replace the tree diagram by a code for deciding who gets the gift. Start with C:

T: C, how must the marbles be selected so that you get the gift?

C: First a white and then a black.

T: So, your code is: White Black.

Have the other students (A, B and D) decide on their codes and write them on the board.

A: Black Black

B: Black White

C: White Black

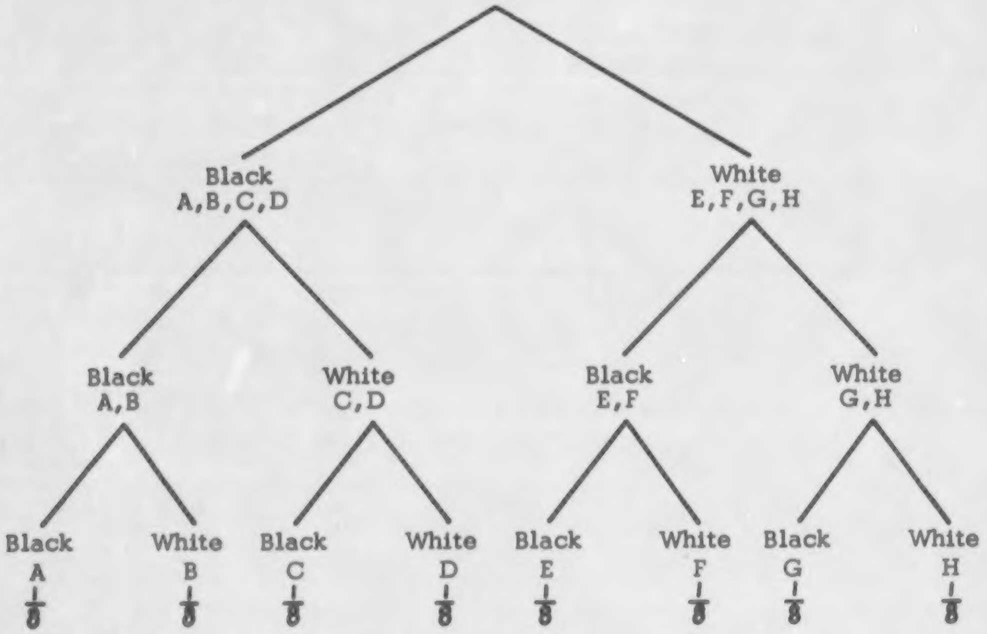
D: White White

T: These codes replace the diagram. (Erase the tree diagram.)

Perform the experiment of selecting one of the four children by choosing two marbles with replacement and using the code.

T: How could we use our two marbles to make a fair choice of one child among eight?

A class discussion should quickly lead to a tree diagram,


Black
A,B,C,D
White
E,F,G,H
Black
A,B
White
C,D
Black
E,F
White
G,H
Black
A
1/8
White
B
1/8
Black
C
1/8
White
D
1/8
Black
E
1/8
White
F
1/8
Black
G
1/8
White
H
1/8

and to the conclusion that each child has 1 chance in 8 of getting the gift. So, this method of picking a marble three times is fair.

1P

Ask students to give you the codes for the eight children, and write them on the board as they are given.

A: Black Black Black
B: Black Black White
C: Black White Black
D: Black White White
E: White Black Black
F: White Black White
G: White White Black
H: White White White

Perform the experiment of picking a marble three times with replacement.

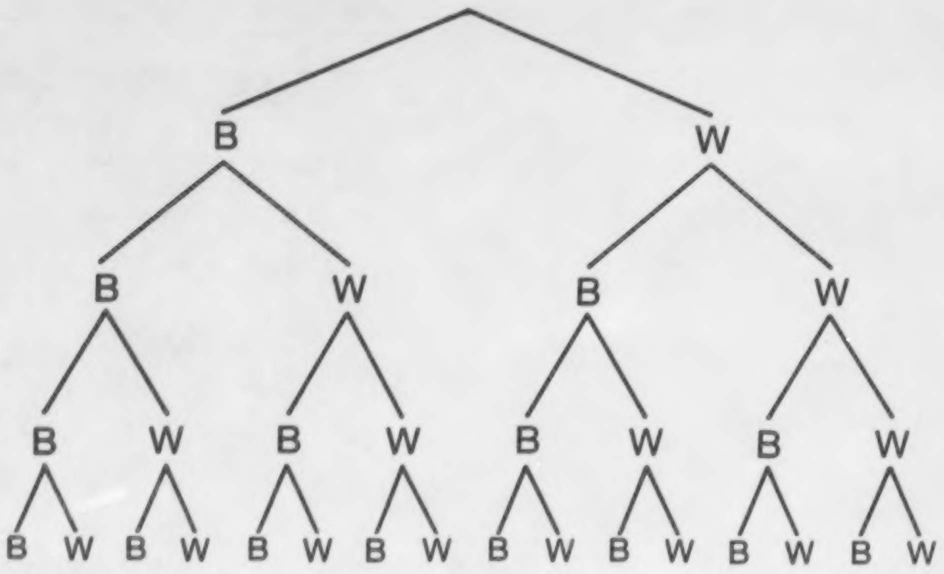
After the first stage, ask who can still get the gift. (If you first pick a white marble, then the answer is A, B, C or D. Otherwise it is E, F, G or H.)

After the second stage, ask who can still get the gift. (Two children, depending on which marble is picked.)

The third stage determines the child who gets the gift.

T: Suppose we pick a marble four times. How many possible outcomes will there be? (16)

With suggestions from your students, draw the tree diagram. ("Black" and "White" are abbreviated "B" and "W", respectively.)



Give some students codes (say John: BBWB and Sue: BWWB) and have them locate where they would be at the end of a path.

Point to the end of a path in the tree and ask the class for the code.

T: How many codes are their with four choices of a color, black or white?

S: 16.

T: How many codes with five choices of a color, black or white?

S: 32.

Accept any reasonable explanation for this answer, including an actual demonstration by extending the tree diagram one more stage.

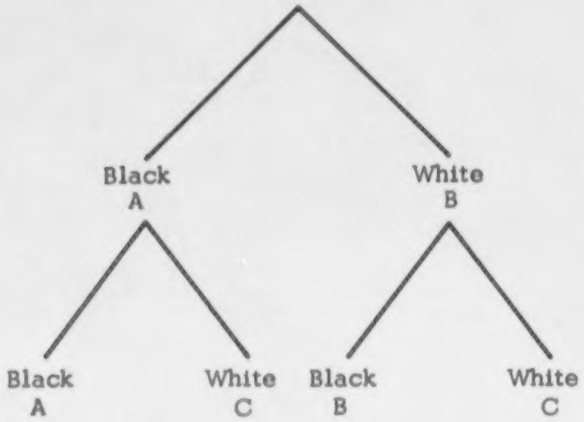
1P

T: Now, suppose the father has a harder problem. He has three children, A, B and C. How can he use the marbles to choose one of the children fairly?

You might get a variety of suggestions. One that was suggested in our development classes, and which you might want your class to consider, is the following.

S: Use the marbles to choose first between A and B. Then use the marbles again to choose between the winner and C.

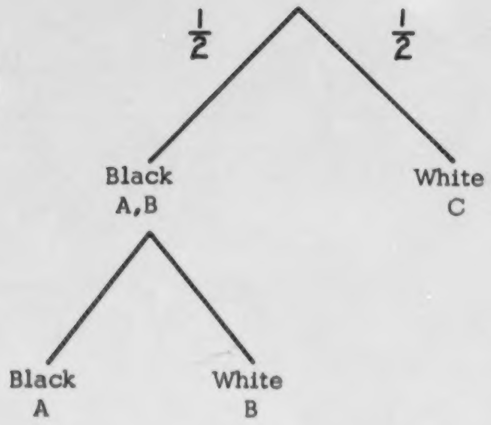
Draw the tree diagram for this suggestion.



T: Is it a fair method?

S: No, C has 2 chances in 4, but A and B each have only 1 chance in 4.

Someone might make a suggestion leading to the following diagram.



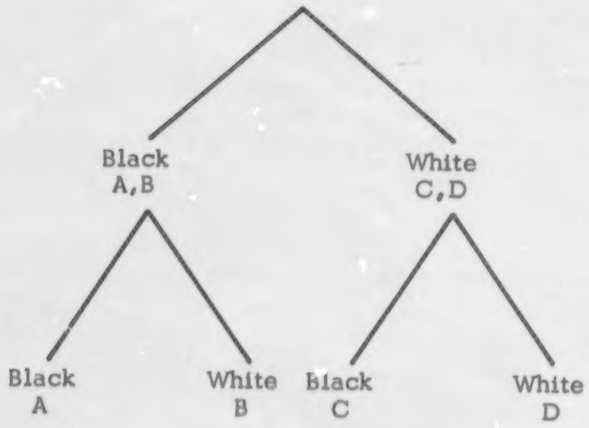
T: Is it a fair method?

S: No, C still has probability $\frac{1}{2}$ of winning.

In this case, you may have to do a little more questioning to have the students recognize that A and B each have probability $\frac{1}{4}$ of getting the gift.

Someone may suggest the following method, but if not you could give some hints.

Go back to the case when there were four children.



T: Suppose we were prepared to choose among four children, and one of the children, D, leaves the room.

S: I see; then if we get White White, then we just start over.

T: Is this fair?

S: Yes, because we will eventually choose one of A, B or C, and they will all have the same chance of being chosen.

If there is sufficient time and interest, and if your class has at most 32 students, you might ask how we can select one member of the class to receive the gift. (It would be more dramatic if you had a small item, such as a candy bar, that would be the "gift" to be won.)

First let the class decide how many times a marble must be selected:

three times if there are at least 5 and at most 8 students.

four times if there are at least 9 and at most 16 students.

five times if there are at least 17 and at most 32 students.

As an example, suppose your class has 27 students. Then the process would require a five-stage tree. At the end of this lesson you will find the 32 code names for picking a marble five times. Copy this page on a sheet of paper and then cut it into 32 individual pieces, one code name on each piece. Distribute one to each student. (If you have 27 students, then five code names will be discarded.) Then perform the experiment of picking a black or white marble five times. After each pick of a marble, write the result on the board (just as you would write a code name) and ask those students who can still win to put up their hands. Each time you pick a marble, fewer students will have their hands up. (Of course, if the "winner" is one of the discarded code names, you will need to start over.)

1P

CODE NAMES

B B B B B
B B B B W
B B B W B
B B B W W
B B W B B
B B W B W
B B W W B
B B W W W
B W B B B
B W B B W
B W B W B
B W B W W
B W W B B
B W W B W
B W W W B
B W W W W

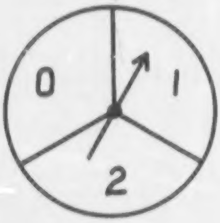
W W W W W
W W W W B
W W W B W
W W W B B
W W B W W
W W B W B
W W B B W
W W B B B
W B W W W
W B W W B
W B W B W
W B W B B
W B B W W
W B B W B
W B B B W
W B B B B

1.2 Probabilities

For this lesson you will need three black and two white marbles. Have them available for Exercise 2.

Exercise 1

Draw a picture of this spinner on the board,

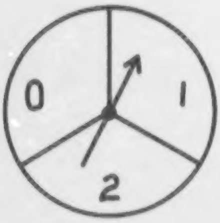


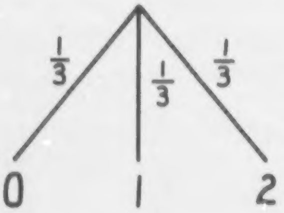
and ask the class to imagine spinning the spinner.

T: What are the possible outcomes? (0, 1 and 2)

What are the probabilities of the outcomes? (1 out of 3 or $\frac{1}{3}$ for each outcome)

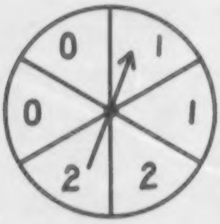
Draw the associated probability tree as students respond to these questions.

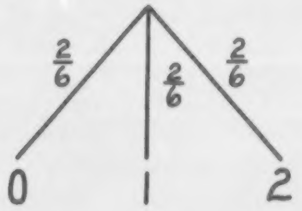




2 P

Repeat these same questions for this spinner and draw the probability tree.

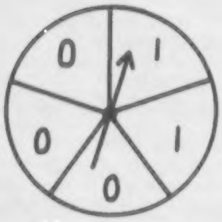


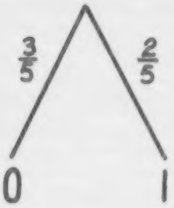


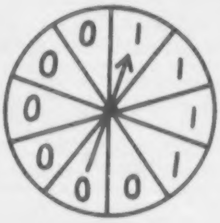
Compare these two spinners and the probabilities of the outcomes. Then ask the students to imagine other spinners that would give the same outcomes with the same probabilities. The result of the discussion should be a lengthy statement of equalities, such as

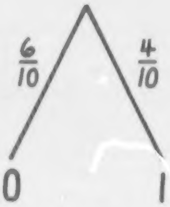
$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \frac{60}{180} = \frac{100}{300} = \frac{10,000}{30,000} = \ldots$$

Repeat the exercise for






0
1
0
1
0
1
0
1
0
0



In this case, you will elicit statements such as

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{12}{20} = \frac{24}{40} = \frac{30}{50} = \frac{150}{250} = \ldots$$

and

$$\frac{2}{5} = \frac{4}{10} = \frac{6}{15} = \frac{8}{20} = \frac{16}{40} = \frac{20}{50} = \frac{100}{250} = \ldots$$

It is not necessary to press the class for an explicit explanation of how these statements are determined. Stop the activity as soon as it begins to drag.

Exercise 2

Take two black and two white marbles in your hands and show the class what you have. Then put your hands behind your back, shake the marbles, and transfer two of them to your right hand. Bring your closed right hand forward. Do not show the class what is in your hand.

T: I picked two of the marbles and they are here in my right hand. How many black marbles could I have in this hand.

S: One or two or none.

T: So the outcome could be 0, 1 or 2 black marbles.

Open your hand so that the class can observe which outcome has occurred. Repeat the experiment several times, until all of the possible outcomes have occurred at least once.

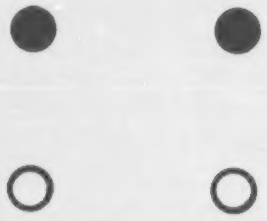
T: What do you think the probability is of getting 0 black marbles in this experiment? 1 black marble? 2 black marbles?

S: They are all $\frac{1}{3}$. There are just three possible outcomes, so each has 1 chance in 3.

T: How can we find out whether you are right?

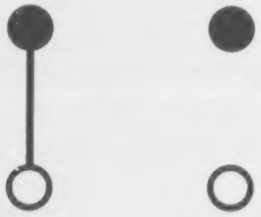
S: Do the experiment a lot of times. Do it 300 times and see if you get about 100 of each outcome.

T: Yes, that's one way to check, but would we ever be sure? Let's try to check by thinking about probabilities. I'll draw a picture of the marbles on the board.



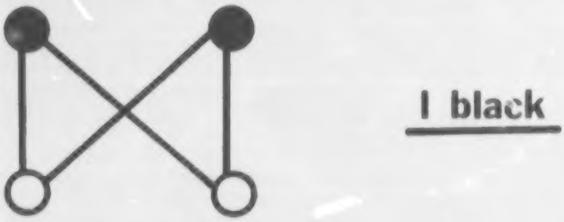
Who can draw a blue cord between two marbles to show a way of getting one black marble; that is, one black and one white marble?

Have a student draw a cord (there are four possible).



T: All right. If the two marbles that are joined with a blue cord are in my right hand, then we have an outcome of 1 black. Any other ways?

Let students draw blue cords until it is agreed that all are drawn.

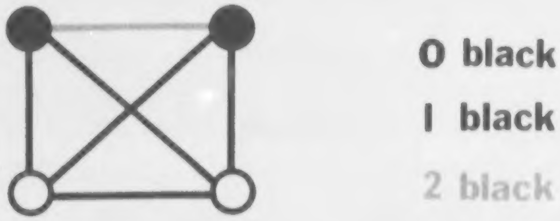


T: Can someone draw yellow cords to show all the ways of getting two black marbles; that is, the outcome 2 black?

Have a student draw the one yellow cord and check with the class that there are no more.

T: Can someone draw red cords to show all the ways of getting no black marbles; that is, the outcome 0 black?

Have a student draw the one red cord and check. Your final picture should look like this.



T: How many cords altogether are there? (6)

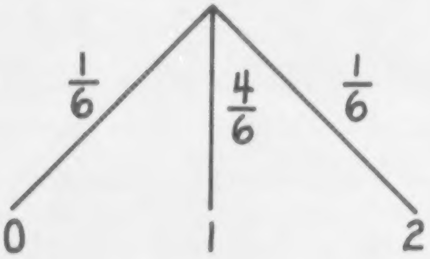
How many red cords? (1)

How many blue cords? (4)

How many yellow cords? (1)

Now, what do you think is the probability of getting 1 black? ($\frac{4}{6}$ or 4 chances out of 6); of getting 0 black? ($\frac{1}{6}$); of getting 2 black? ($\frac{1}{6}$)

Draw a probability tree for this experiment.



$\frac{1}{6}$
$\frac{4}{6}$
$\frac{1}{6}$
0
1
2

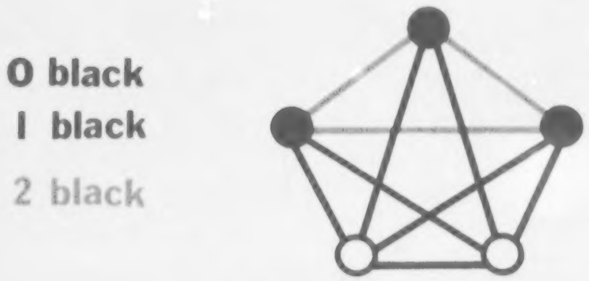
T: If we do this experiment 300 times, do you think we would get about 100 of each outcome?

S: No, we would get 0 black about 50 times, about once in every six times.

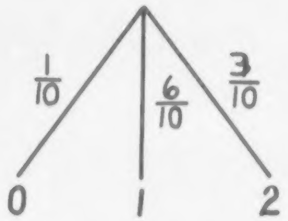
S: And we would get 2 black about 50 times, and 1 black about 200 times.

Repeat this exercise using three black and two white marbles.

The picture and probability tree for this experiment are:



0 black
1 black
2 black



$\frac{1}{10}$
$\frac{6}{10}$
$\frac{3}{10}$
0
1
2

Have the diagrams for the two experiments side by side on the board.

T: Suppose you are playing a game using one of these two experiments. You will win if you get 1 black. Which experiment would you prefer to use?

S: The second one, because there the probability of 1 black is bigger; $\frac{6}{10}$ is more than $\frac{4}{6}$.

T: Can you convince us that $\frac{6}{10}$ is more than $\frac{4}{6}$?

Let students try to explain why they think one probability or the other is the larger, encouraging them to think in terms of 4 chances in 6 and 6 chances in 10. If no student suggests it, ask for other ways of thinking of 4 chances in 6 and of 6 chances in 10 leading to

$$\frac{4}{6} = \frac{8}{12} = \frac{12}{18} = \frac{16}{24} = \frac{20}{30} = \frac{24}{36} = \dots$$

and

$$\frac{6}{10} = \frac{12}{20} = \frac{18}{30} = \frac{24}{40} = \dots$$

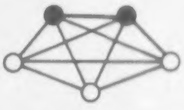
From this it can be seen that 4 chances in 6 is the same as 20 chances in 30, and that 6 chances in 10 is the same as 18 chances in 30. Clearly, 20 chances in 30 is a larger probability than 18 chances in 30. Therefore, the first experiment would be better to win with 1 black.

NOTE: There might be a temptation here to speak of "common denominators" or even "the least common denominator". It would be better not to use this language, but rather to guide the students to discover that the problem of comparing probabilities is related to the problem of finding common multiples.

Worksheets P01 and P02 are available for the rest of the lesson.

Name ______________________ **P 01** *

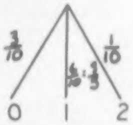
Here are two black and three white marbles.



PICK ANY TWO MARBLES

Draw red cords for all the ways of getting two white marbles.
Draw blue cords for all the ways of getting one black and one white marble.
Draw black cords for all the ways of getting two black marbles.

How many red cords? 3
How many blue cords? 6
How many black cords? 1
How many cords in all? 10

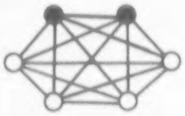


What is the probability of getting 0 black marbles? 3/10
What is the probability of getting 1 black marble? 6/10 = 3/5
What is the probability of getting 2 black marbles? 1/10

Write these probabilities in the tree picture.

Name ______________________ **P 02** **

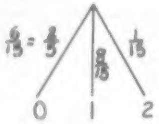
Here are two black and four white marbles.



PICK ANY TWO MARBLES

Draw red cords for all the ways of getting two white marbles.
Draw blue cords for all the ways of getting one black and one white marble.
Draw black cords for all the ways of getting two black marbles.

How many red cords? 6
How many blue cords? 8
How many black cords? 1
How many cords in all? 15



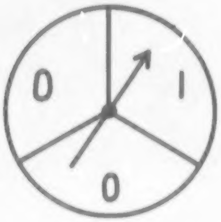
What is the probability of getting 0 black marbles? 6/15 = 2/5
What is the probability of getting 1 black marble? 8/15
What is the probability of getting 2 black marbles? 1/15

Write these probabilities in the tree picture.

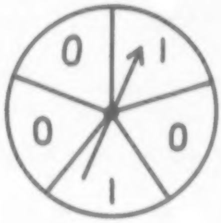
1.3 Fair Game? #1

Exercise 1

Draw pictures of two 0,1 - spinners on the board.



A

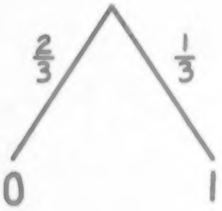


B

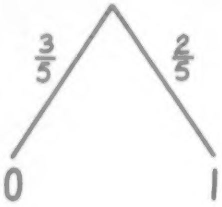
T: Suppose you have these two spinners, and you are playing a game. You will be the winner if the spinner stops on "1". Which of these spinners would you prefer to use?

Let the students discuss this problem. The discussion should involve the probabilities of 0 and 1 with the respective spinners. Draw the probability trees for these spinners next to them on the board.

A



B



Some students may be able to explain that B is preferable because the probability of 1 is larger; that is, that $\frac{2}{5}$ is more than $\frac{1}{3}$. Listen to their suggestions; they might suggest considering other spinners similar to A and B with the same probabilities for 0 and 1 (as in Section 1.2, Exercise 1). Otherwise, you could ask them to consider what they think would occur if each spinner was spun many times (say 30 or 60 or 120). From this discussion you should generate statements like these which you should write on the board.

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \frac{5}{15} = \frac{10}{30} = \ldots$$

$$\frac{2}{5} = \frac{4}{10} = \frac{6}{15} = \frac{8}{20} = \frac{12}{30} = \ldots$$

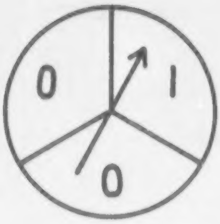
S: 2 chances in 5 is the same as 6 chances in 15 (or 12 chances in 30), and 1 chance in 3 is the same as 5 chances in 15 (or 10 chances in 30). So $\frac{2}{5}$ is more than $\frac{1}{3}$.

Students may compare the probabilities using any common denominator. It is not necessary that they use the least common denominator.

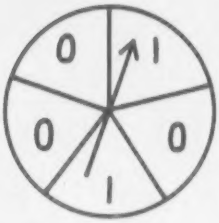
Repeat the exercise comparing $\frac{3}{4}$ and $\frac{5}{6}$.

Exercise 2

Draw on the board pictures of the spinners used at the beginning of Exercise 1.



A



B

Tell a story about a boy, Bruce, and his two brothers, Allen and Charles. Bruce likes to make up games for the three boys to play. One day he came with these two spinners. Because there are three boys and only two outcomes for each spinner, he made up a new game which would use both spinners.

Bruce's game: First spin spinner A and then spin spinner B. Add the two numbers you get.

T: What could the sum be?

S: 0, if you get 0 with both spinners.
1, if you get 0 with one of the spinners and 1 with the other.
2, if you get 1 with both spinners.

T: Good, there are three possibilities, so the three boys can play. Bruce's idea was:

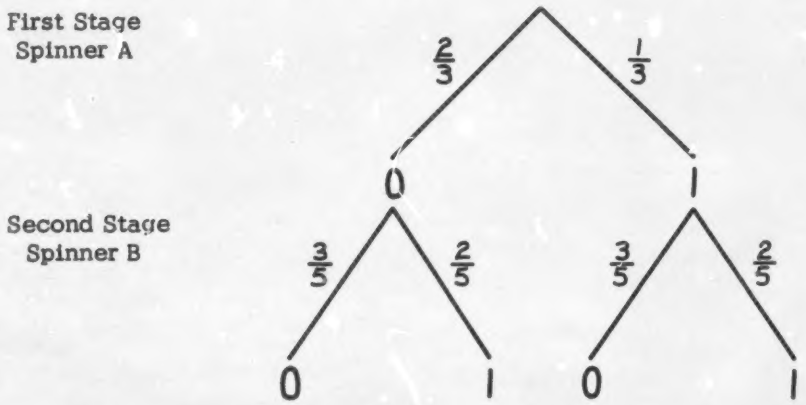
if the sum is 0, then Allen wins;
if the sum is 1, then Bruce wins;
and if the sum is 2, then Charles wins.

3P

Allen and Charles are always suspicious of their brother's games, so they wondered whether or not it was a fair game. Do you think Bruce has invented a fair game?

Let the students discuss the game. Some may suggest that it is fair because there are three possibilities and each boy has 1 chance in 3. Others may recognize immediately that a sum of 2 has less chance than a sum of 0, but not be able to compare the chances for sums of 1 and 2. Ask the students to try to find a way of deciding whether or not it is a fair game.

Suggest to the students that we could draw a probability tree for this game, and draw it on the board.

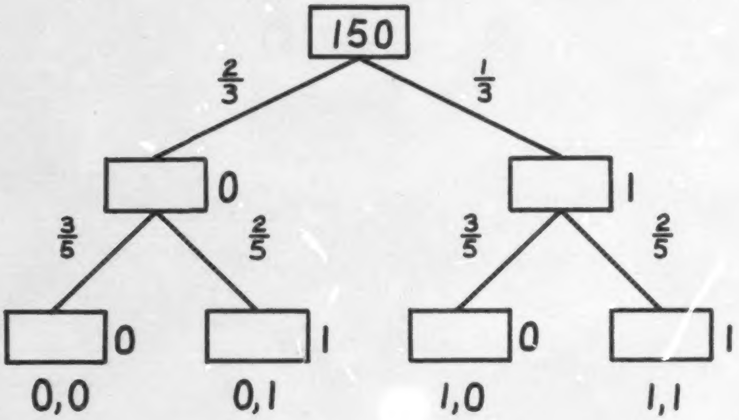

First Stage
Spinner A
$\frac{2}{3}$
$\frac{1}{3}$
0
1
Second Stage
Spinner B
$\frac{3}{5}$
$\frac{2}{5}$
$\frac{3}{5}$
$\frac{2}{5}$
0
1
0
1

Trace paths in the tree and have the students describe the outcomes as

0,0 0,1 1,0 1,1

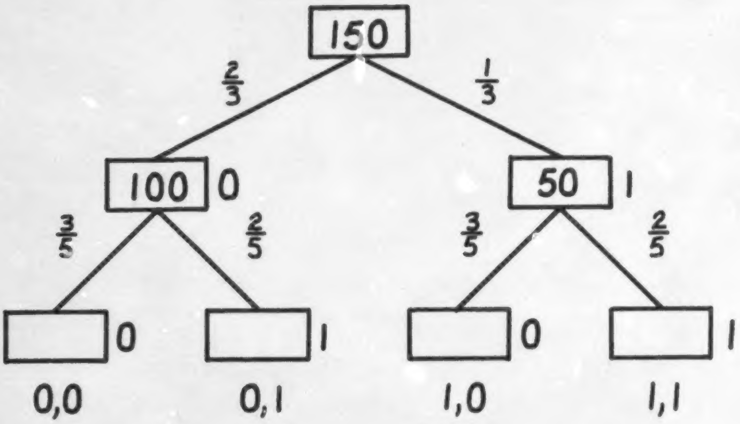
T: Suppose the three boys played the game 150 times.

Draw boxes in your picture as shown below, and enter 150 in the top box.


150
2/3
1/3
0
1
3/5
2/5
3/5
2/5
0
1
0
1
0,0
0,1
1,0
1,1

T: First use spinner A. About how many times do we expect to get 0? (100); to get 1? (50)

Fill in the appropriate boxes.

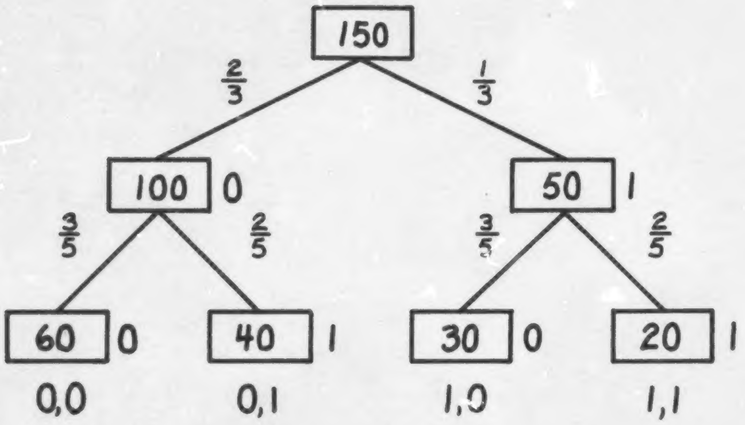

150
2/3
1/3
100
0
50
1
3/5
2/5
3/5
2/5
0
1
0
1
0,0
0,1
1,0
1,1

T: Next, use spinner B.

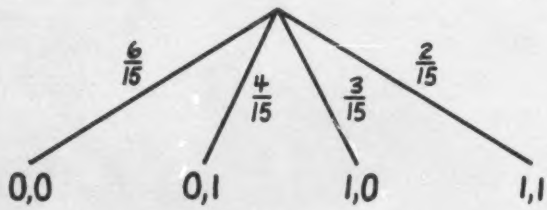
Of these 100 trials (point to the appropriate box), about how many times do we expect to get 0? (60); to get 1? (40)

Of these 50 trials (point to the appropriate box), about how many times do we expect to get 0? (30); to get 1? (20)

Fill in the boxes as answers are given.

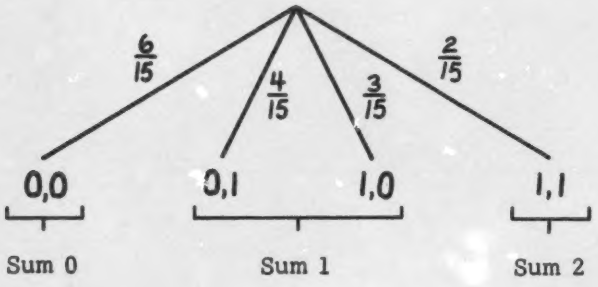

150
2/3
1/3
100
0
50
1
3/5
2/5
3/5
2/5
60
0
40
1
30
0
20
1
0,0
0,1
1,0
1,1

Then draw a short probability tree for this game, asking students to provide the probabilities of each outcome.


6/15
4/15
3/15
2/15
0,0
0,1
1,0
1,1

Students might suggest $\frac{60}{150}$ as the probability of 0,0. This answer should be accepted. You could then ask for other ways to write "60 chances out of 150" before writing the probabilities on the tree, but it is not essential.

T: Now, it should be easier to decide whether this game is fair. Remember, Allen wins if the sum is 0; Bruce wins if the sum is 1; Charles wins if the sum is 2.



Of the 150 games, about how many will Allen win? (60)

What is Allen's probability of winning? (60 in 150 or $\frac{6}{15}$)

Of the 150 games, about how many will Bruce win? (70)

What is Bruce's probability of winning? (70 in 150 or $\frac{7}{15}$)

Of the 150 games, about how many will Charles win? (20)

What is Charles' probability of winning? (20 in 150 or $\frac{2}{15}$).

S: The game is not fair. Bruce has the best chance of winning and Charles has the worst.

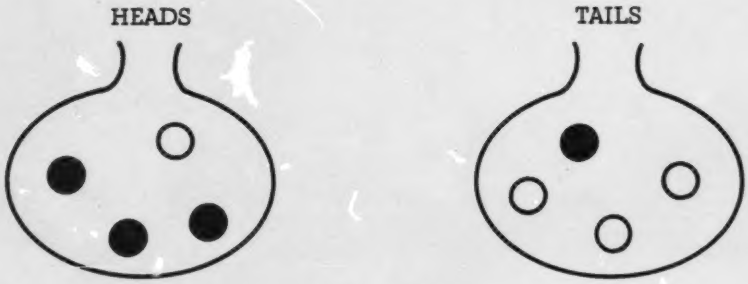
T: Yes; that is what Allen and Charles found out too, and they weren't very happy with Bruce's game.

If this exercise goes well and there is still time (at least 15 minutes), you could introduce another game situation to analyze. Continue the story by explaining that Allen and especially Charles were not very pleased with this game. So Bruce invented another game, this time using black and white marbles.

Bruce has four black and four white marbles. If they would pick one marble, there would be only two outcomes and he wants a game for three people. Bruce also likes to make the game more exciting. So, he puts the marbles into two bags. In one bag, he puts three black and one white. In the other, he puts three white and one black.

Bruce's new game: Flip a coin. If it comes up heads, pick two marbles from the first bag. If it comes up tails, pick two marbles from the second bag.

Draw this picture as you describe the game.

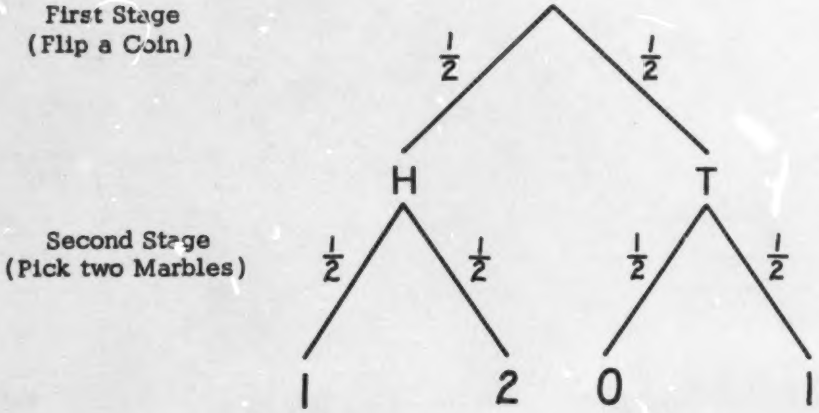


If the outcome is 0 black, then Allen wins.

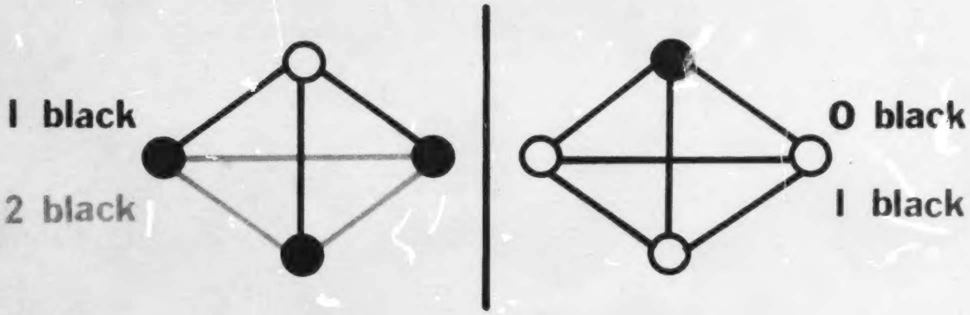
If the outcome is 1 black, then Bruce wins.

If the outcome is 2 black, then Charles wins.

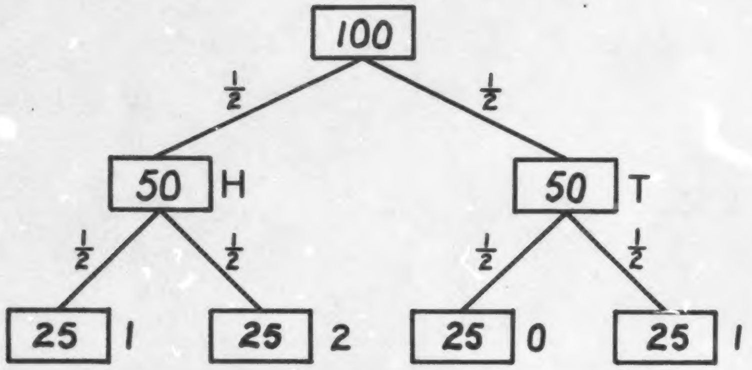
Let the students react to this new game. They should be on guard for its being unfair. In any case, ask them how to decide whether the game is fair. The discussion should lead to a two stage probability tree.


First Stage
(Flip a Coin)
$\frac{1}{2}$
$\frac{1}{2}$
H
T
Second Stage
(Pick two Marbles)
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
1
2
0
1

To find the probabilities in the second stage, you could use cord pictures like those in Section 1.2.


1 black
2 black
0 black
1 black

Then, lead a discussion of what to expect if the game were played say 100 or 200 times.


100
$\frac{1}{2}$
$\frac{1}{2}$
50 H
50 T
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
25 1
25 2
25 0
25 1

The students should conclude that the game is not fair.

Allen's probability of winning is $\frac{1}{4}$ (25 in 100).

Bruce's probability of winning is $\frac{1}{2}$ (50 in 100).

Charles' probability of winning is $\frac{1}{4}$ (25 in 100).

In fact, Bruce's chances of winning are even better in this game than the last.

If the class is interested in these games, you could ask them to try to find a fair game for the three boys using either the two spinners or white and black marbles.

1.4 Fair Game? #2

For this lesson, you will need one red die and one black die (or any two different color dice, changing the colors in the lesson appropriately).

Exercise 1

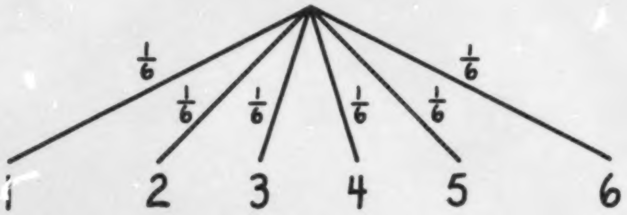
Show the class that you have two different color dice. Let the students describe a die - it is a cube; it has six faces; it is symmetrical; each face has some dots on it, either 1 or 2 or 3 or 4 or 5 or 6. You might ask some questions which elicit such a description as well as establish the language; "die" is singular and "dice" is plural. When you roll a die, the result (outcome) is the number of dots on the top face. Do this several times and have a student read the outcome.

T: What are the possible outcomes when I roll one die? (1,2,3,4,5,6)

Do you think one of these outcomes has a better chance of occurring than another? (No)

What is the probability of 6? ($\frac{1}{6}$); of 1? ($\frac{1}{6}$); of 5? ($\frac{1}{6}$); etc.

Draw a probability tree for the experiment of rolling one die.



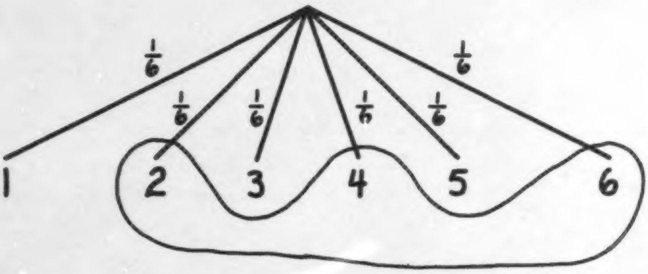
T: What are the chances that the outcome will be an even number?

S: 3 chances in 6.

T: So, what is the probability of an even number?

S: $\frac{3}{6}$ or $\frac{1}{2}$.

To help the students see that one-half of the outcomes are even, you could draw a string around the even number outcomes.

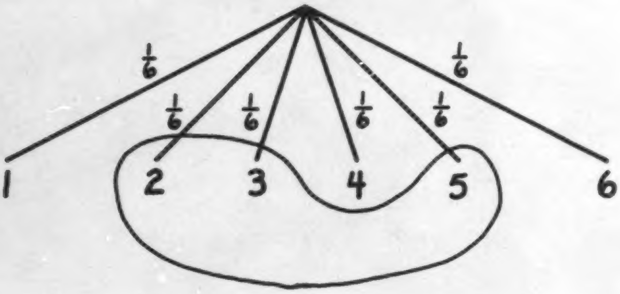

$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
1
2
3
4
5
6

Erase the string.

T: What is the probability of getting a prime number?

S: $\frac{1}{2}$, because half of the outcomes are prime; 2, 3 and 5.

Draw a string around the prime number outcomes.

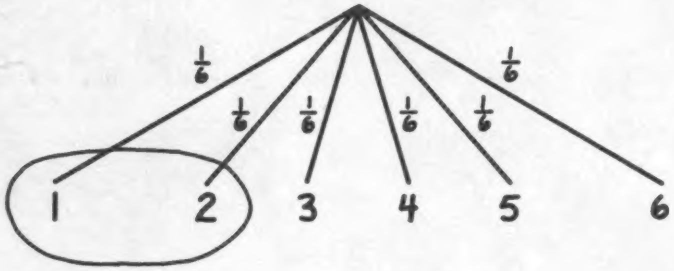

$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
1
2
3
4
5
6

Erase the string.

T: What is the probability of getting at most 2?

S: $\frac{2}{6}$ (or $\frac{1}{3}$).

Here, you may have to review what "at most" means. Do this as you draw a string around these outcomes.



Continue the exercise with a few more questions.

What is the probability of getting at least 5? ($\frac{2}{6}$ or $\frac{1}{3}$)

What is the probability of getting less than 5? ($\frac{4}{6}$ or $\frac{2}{3}$)

What is the probability of getting a divisor of 12? ($\frac{5}{6}$)

Exercise 2

Remind the class of the three brothers, Allen, Bruce, and Charles, and how Bruce liked to invent games of chance for the three boys to play.

Bruce came to his brothers with a die and proposed the following game.

Roll the die and if the outcome is

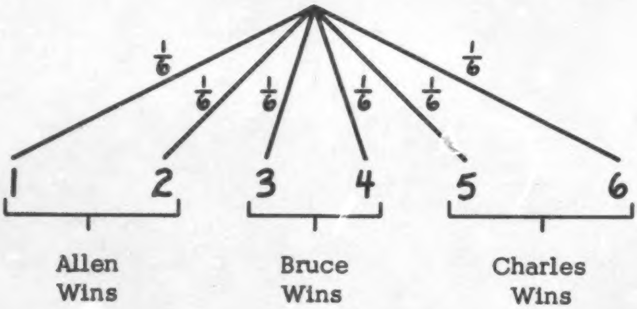
1 or 2, then Allen wins;
3 or 4, then Bruce wins;
5 or 6, then Charles wins.

T: Do you think this game is fair?

S: Yes, each boy has 2 chances out of 6 to win.

S: The probability of winning is $\frac{1}{3}$ for each boy.

Draw the probability tree and arrive at general agreement that, indeed, Bruce has finally proposed a fair game.



T: As you can imagine, Bruce's brothers were happy to finally have a fair game. But Bruce quickly got bored of such a simple game; he liked more complicated games. Bruce said he had a much better game using two dice, one red and one black. Before I tell you Bruce's new game, let's record the outcomes when we roll two dice.

Roll the dice several times and have students report the outcomes. Record the outcomes as they are reported; for example,

Red 6, Black 3

Red 4, Black 3

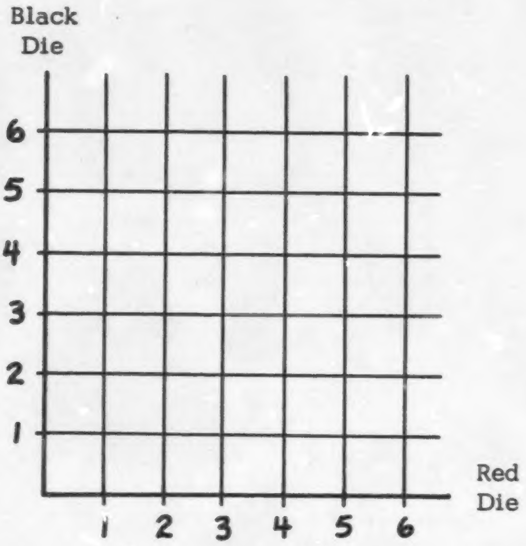
Red 3, Black 4

etc.

Be sure they report what number is on which die.

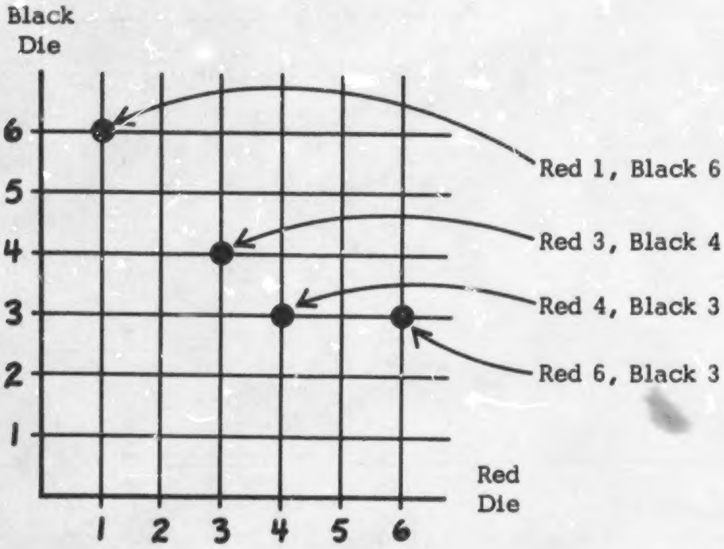
T: We need a way to represent all the possible outcomes.

Some students may give you suggestions, but in any case suggest they consider this picture.

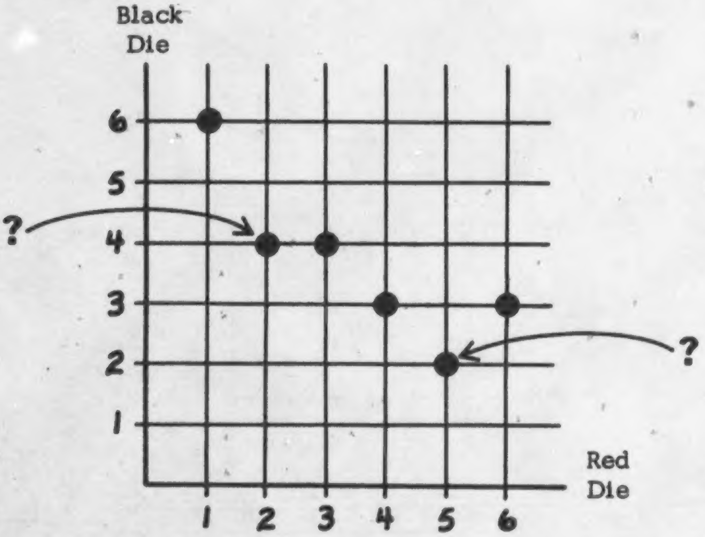


T: Where do you think I'll put a dot for the outcome, Red 6, Black 3?

Let students make guesses, until someone finds the right point. Repeat this question for several more outcomes (those obtained in your rolls of the two dice).


Black Die
6
5
4
3
2
1
Red 1, Black 6
Red 3, Black 4
Red 4, Black 3
Red 6, Black 3
Red Die
1
2
3
4
5
6

After seeing how to represent a few outcomes, most students will see the pattern. Vary the questioning by drawing other dots and asking for the corresponding outcomes.


Black Die
6
5
?
4
3
2
?
1
Red Die
1
2
3
4
5
6

For these two dots, students should respond "Red 2, Black 4" and "Red 5, Black 2".

T: How many possible outcomes are there when one rolls two dice?

S: 36, because 6 x 6 = 36.

Draw in the remaining dots.

NOTE: If some of your students would benefit from additional work with this grid, you could organize a Grid Tic Tac Toe game during free time. The grid used in such a game can be extended beyond this six by six one.

Grid Tic Tac Toe: Divide a group into two teams, A and B. Taking turns, the teams suggest outcomes (say, Red 3, Black 5) and the teacher (or recorder, if you have a good student to act as recorder) marks that dot with an "X" (for Team A) or a "O" (for Team B). The object is for one team to get four dots in a row either horizontally, or vertically, or diagonally.

T: Now I can tell you about Bruce's new game. The rules are:

Roll two dice and add the numbers they show.

If the sum is

3, 4 or 5, then Allen wins;

6, 7 or 8, then Bruce wins;

9, 10 or 11, then Charles wins.

What do you think about Bruce's new game?

S: What happens if the sum is 2 or 12?

T: That's a good question. What outcome gives a sum of 2? (Answer: Red 1, Black 1); a sum of 12? (Answer: Red 6, Black 6) Are there other numbers that could be the sum and that Bruce didn't mention in his game?

As the students check what numbers the sum could be, write these numbers on the board.

$$\cancel{2}\quad \underbrace{3\quad 4\quad 5}_{\text{Allen}}\quad \underbrace{6\quad 7\quad 8}_{\text{Bruce}}\quad \underbrace{9\quad 10\quad 11}_{\text{Charles}}\quad \cancel{12}$$

T: In Bruce's game, if the sum is 2 or 12, then the boys roll the dice again (cross out "2" and "12" in your list, and indicate which boy wins for the other numbers).

Do you think this is a fair game?

Let the students comment on this game. Some students may think it is fair because each boy wins with three of the remaining nine possible sums. Others may recognize that not all of the sums are equally likely. You could suggest that both Allen and Charles were very suspicious; they know that Bruce likes to trick them into playing unfair games.

Direct the class' attention back to the grid on the board.

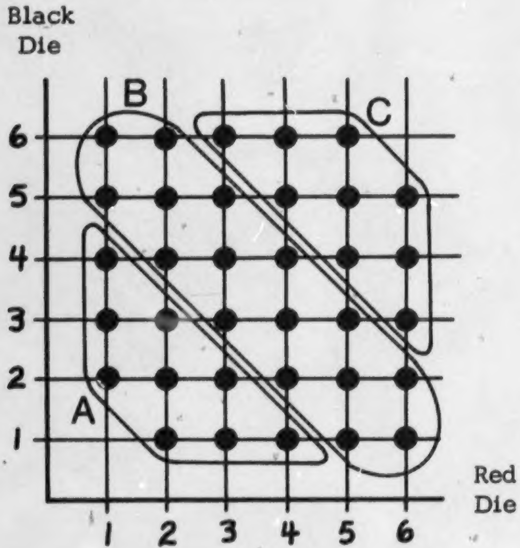
T: How many of these 36 outcomes are the boys using in this game? (34) Is one of these just as likely as another? (Yes, the dice are symmetrical, or the dice are not lopsided, etc.)

Now let's decide which of these outcomes will make Allen the winner.

Let students suggest outcomes where the sum is 2, then where the sum is 3, then where the sum is 4. Individual students might begin to recognize that all outcomes with the same sum lie on a diagonal of the grid, but do not make this an objective of the activity. When all such outcomes have been found, draw a string around them and label it "A" for Allen.

T: What outcomes will make Bruce the winner? Charles?

In each case, go through the same questioning as before, and draw strings around the corresponding points, labeling the strings "B" for Bruce and "C" for Charles. Now the grid looks like this.



T: What do you think? Is it a fair game?

S: Bruce has the best chance again!

S: There are 34 possibilities, and Allen and Charles each has only 9 chances in 34. But Bruce has 16 chances in 34.

T: What are the probabilities of winning for the three boys?

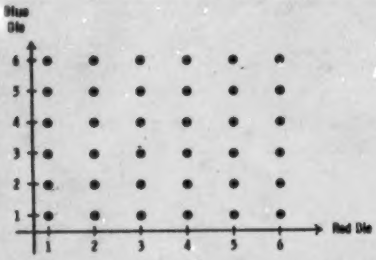
S: The probability that Bruce wins is $\frac{16}{34}$.

S: The probability that Allen wins is only $\frac{9}{34}$ and the same for Charles.

Distribute Worksheet P03 and explain that the picture shows all the outcomes when two dice are rolled (none is omitted here). Let the class agree that there are a total of 36 possible outcomes, and explain that the first part of the worksheet asks them to write the probability of each of the possible sums of the two numbers in an outcome. As you watch students working, you might help those having difficulties by asking them to draw a string around all the outcomes with a particular sum. In filling in the table of probabilities, do not insist upon reduction of fractions. It is best to keep the probabilities in the form "$\frac{1}{36}$", "$\frac{2}{36}$", "$\frac{3}{36}$", "$\frac{4}{36}$", etc., so that they can be used easily in the rest of the questions.

Worksheet P04 is on the back of the page for those who finish P03.

Name ______________________ **P 03** *



Complete the table.

Sum	2	3	4	5	6	7	8	9	10	11	12
* Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Which sum has the greatest probability? 7

What is the probability that the sum is less than 4? $\frac{3}{36} = \frac{1}{12}$

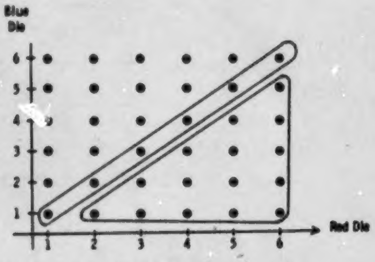
What is the probability that the sum is at most 4? $\frac{6}{36} = \frac{1}{6}$

What is the probability that the sum is greater than 9? $\frac{6}{36} = \frac{1}{6}$

What is the probability that the sum is at least 9? $\frac{10}{36} = \frac{5}{18}$

*Some of these fractions can be reduced; e.g., $\frac{2}{36} = \frac{1}{18}$; $\frac{3}{36} = \frac{1}{12}$; $\frac{4}{36} = \frac{1}{9}$; $\frac{6}{36} = \frac{1}{6}$. Accept any correct answer whether reduced or not.

Name ______________________ **P 04** **



Refer to the Table on P03

What is the probability that the sum is even? $\frac{18}{36} = \frac{1}{2}$

What is the probability that the sum is odd? $\frac{18}{36} = \frac{1}{2}$

What is the probability that the sum is prime? $\frac{15}{36} = \frac{5}{12}$

What is the probability that the sum is a divisor of 12? $\frac{12}{36} = \frac{1}{3}$

Draw a string around the outcomes for which the same number is on both the blue and red dice. The string is drawn in red above.

What is the probability that the same number is on both dice? $\frac{6}{36} = \frac{1}{6}$

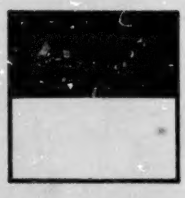
Draw a string around the outcomes for which the number on the red die is greater than the number on the blue die. The string is drawn in blue above.

What is the probability that the number on the red die is greater than the number on the blue die? $\frac{15}{36} = \frac{5}{12}$

1.5 Fair Game? #3

Exercise 1

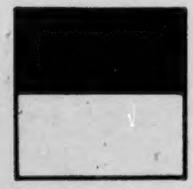
Draw this picture on the board.



T: Suppose a tiny flea is going to land on this square and it does not prefer any color; it lands at random. What is the probability the flea will land on red? on white?

S: The probability of red is $\frac{1}{2}$ and of white is $\frac{1}{2}$.

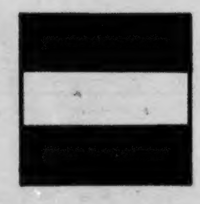
Write these probabilities on the board next to the square.



$r = \frac{1}{2}$

$w = \frac{1}{2}$

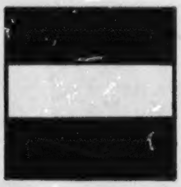
Next draw this square.



T: Suppose the flea were going to land on this square. What is the probability of red? of white?

S: Red $\frac{2}{3}$ and white $\frac{1}{3}$.

Write these probabilities on the board next to the square.



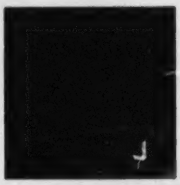
$r = \frac{2}{3}$

$w = \frac{1}{3}$

T: In this case, imagine that we were going to observe the flea land, say 60 times. What would you predict would ha[illegible]n?

S: It would land about 40 times on red and about 20 times on white.

Next draw a completely red square.



$r = 1$

$w = 0$

T: If the flea were going to land on this square, then what would be the probability of red? of white?

S: Red is 1 and white is 0.

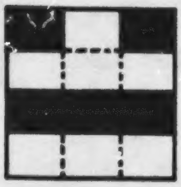
Write these probabilities on the board next to the square.

T: In this case, if we could observe the flea make many landings what would we expect to happen?

S: It will always land on red.

T: Yes, we are certain it will land on red and not on white.

Next draw this square.



$r = \frac{5}{12}$

$w = \frac{7}{12}$

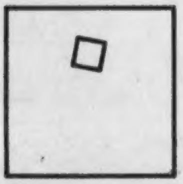
NOTE: As you make this drawing, division of the square into 12 pieces should be evident to the students watching. However, when you shade the red part and the white part, the division lines should not be easily visible.

T: Now, what are the probabilities?

S: Red $\frac{5}{12}$ and white $\frac{7}{12}$.

Draw an unshaded square on the board.

T: Suppose you know only that for this square the probability of the flea landing on red is $\frac{2}{3}$ and on white is $\frac{1}{3}$. You do not know where the red part is or where the white part is. Now imagine that we cut out a small square. (Illustrate a square cutout on your picture.)



$r = \frac{2}{3}$

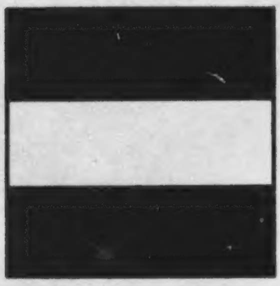
$w = \frac{1}{3}$

T: What can you say about the small cutout square?

S: It could be all red or all white.

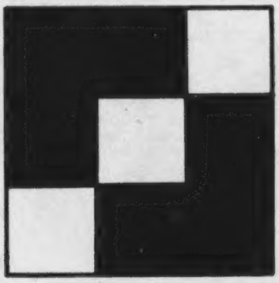
S: It might be part red and part white.

Accept any response such as these two, but encourage the students to think about the small square being like the bigger square; i.e., $\frac{2}{3}$ red and $\frac{1}{3}$ white. You should illustrate some possibilities for the big square by showing successively the following three colored squares - these pages are provided at the end of this lesson. Each time ask for the probabilities of red and of white.



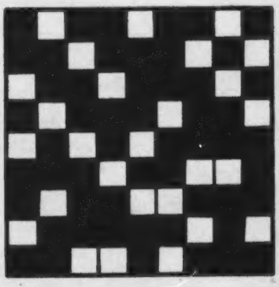
$r = \frac{2}{3}$

$w = \frac{1}{3}$



$r = \frac{6}{9} = \frac{2}{3}$

$w = \frac{3}{9} = \frac{1}{3}$



$r = \frac{54}{81} = \frac{2}{3}$

$w = \frac{27}{81} = \frac{1}{3}$

The better "mixed" the big square is, the better chance that the small square is similar to it. In problems like this one, we will assume the "mixture" is homogeneous, well mixed. That is, if we consider only part of a certain "mixture", then we assume that part has the same proportions as the original "mixture". For example, suppose we have a cup of coffee (maybe 7 oz.) and put in 1 oz. of cream so that we then have 8 oz. of the coffee with cream mixture. We stir well the mixture and then agree that any part of the cup of coffee is also $\frac{7}{8}$ coffee and $\frac{1}{8}$ cream.

To illustrate the assumption of homogeneity for your students, you could suggest several examples.

1. Coffee with cream.
2. Milk with chocolate syrup - chocolate milk.
3. Rippled ice-cream.

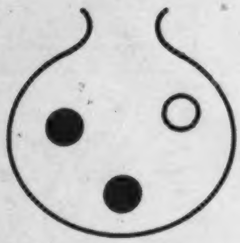
T: We could think of the square as a carton of strawberry ripple ice-cream. It is $\frac{2}{3}$ strawberry (red) and $\frac{1}{3}$ vanilla (white). Then, what can we say about a scoop of ice-cream from this carton?

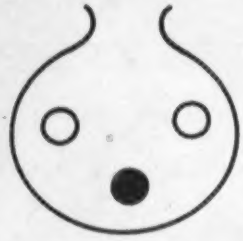
S: It is about $\frac{2}{3}$ strawberry and $\frac{1}{3}$ vanilla.

T: Okay. Now we are going to see how these ideas can help us solve a problem involving probabilities.

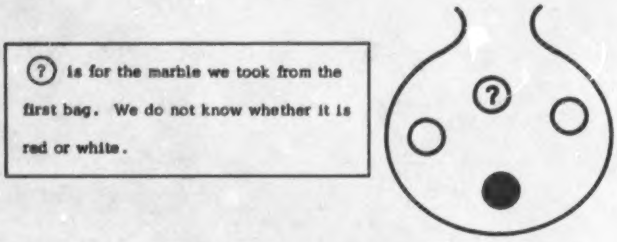
Exercise 2

T: Do you remember the three brothers, Allen, Bruce, and Charles, who liked to play games? One day Charles was not playing, so Bruce suggested a game for two people. (Draw a picture of the situation as you describe it.) Take two bags of marbles. In the first bag, there are two red marbles and one white marble. In the second bag, there are two white marbles and one red marble.


First Bag


Second Bag

T: The game is played like this - take one marble from the first bag and without looking put it into the second bag. Then mix up the marbles in the second bag and take out one.


ⓘ is for the marble we took from the first bag. We do not know whether it is red or white.
?

Second Bag

One boy wins if the marble is red; the other if it is white. What do you think about this game? What do you think are the probabilities for red and for white?

Let the students discuss the game for a couple minutes. You may get a variety of responses. For example,

S: The game is fair. The probabilities are $\frac{1}{2}$ for red and $\frac{1}{2}$ for white.

T: Why?

S: Because there are the same number of red and white marbles in the two bags together. (Or if you put a red marble into the second bag then there are 2 red marbles and 2 white marbles to choose from.)

S: The game is not fair. The probabilities are $\frac{1}{4}$ for red and $\frac{3}{4}$ for white.

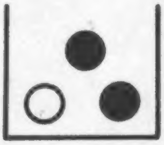
T: Why?

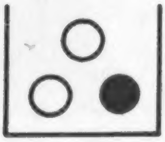
S: Because, if you put the white marble into the second bag, then there are 1 red and 3 white marbles to choose from.

In these cases, tell the students they have good ideas but perhaps they are not thinking about the entire game.

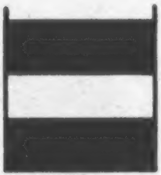
A student might suggest using a probability tree - see the note at the end of this lesson for a description. In this case, accept the suggestion and present it first, but then return to the following way of thinking about the problem.

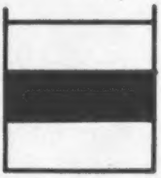
T: Let's think about the situation. We can think about the marbles like scoops of ice-cream. In the first carton we put two scoops of strawberry ice-cream (two red marbles) and one scoop of vanilla (one white marble).


First


Second

Then we mix them (like we were making strawberry ripple ice-cream).

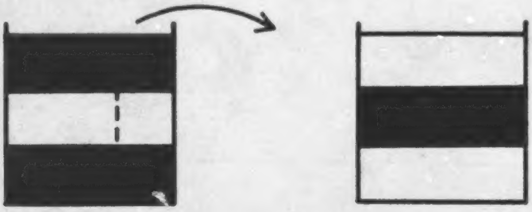

First Carton


Second Carton

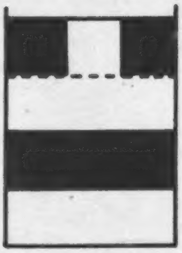
I draw it this way only to remind us that the first carton is $\frac{2}{3}$ strawberry and $\frac{1}{3}$ vanilla, and that the second carton is $\frac{1}{3}$ strawberry and $\frac{2}{3}$ vanilla. However, we will think about them as being well mixed.

Now we will take one scoop from the first carton and put it into the second carton. That is like taking one marble from the first bag and putting it into the second bag. Remember, the scoop of ice-cream we take from the first carton is assumed to be $\frac{2}{3}$ strawberry and $\frac{1}{3}$ vanilla because the ice-cream is mixed.

On the board you should first illustrate taking out one scoop from the first carton. (The first carton has three scoops, so one scoop is $\frac{1}{3}$ of the carton.)



Next illustrate putting the one scoop of ice-cream into the second carton.



T: Now, can you tell me the probabilities for red and white?

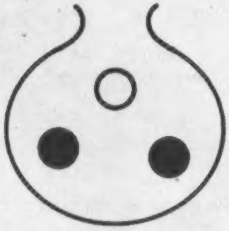
S: Red is $\frac{5}{12}$ and white is $\frac{7}{12}$.

T: Do you think this marble game is fair?

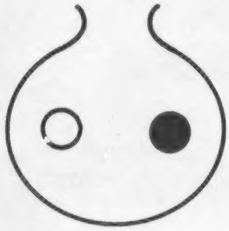
S: No, the probabilities for red and for white are not equal.

Exercise 3

T: When Bruce suggested this game, he, of course, proposed that he win if the result was white and that Allen win if it were red. But Allen quickly thought about the situation and suggested that they change the game by taking one of the white marbles out of the second bag. He wanted to play the same game with these bags of marbles.



First Bag



Second Bag

T: Now, what do you think about this game? Is it fair?

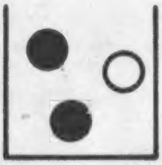
If you did not present the probability tree in the Note on Exercise 2 first, do so now as an alternative way to solve the problem.

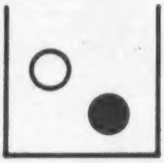
Let the students have several minutes to think about this new game.

S: This game is also not fair. This time the probability for red is $\frac{5}{9}$ and for white is $\frac{4}{9}$.

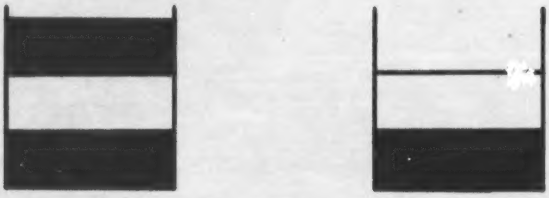
Ask students to explain their solutions. You are likely to receive an explanation similar to this one.

S: The first bag is like a carton with two scoops of strawberry ice-cream and one scoop of vanilla ice-cream. The second bag is like a carton with one scoop of each kind.



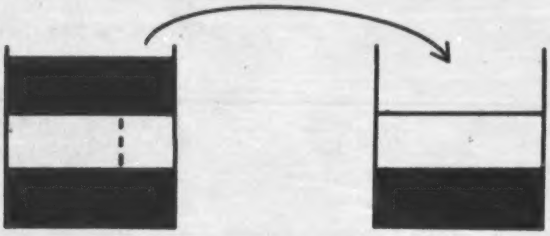


Draw pictures on the board as a student gives an explanation, helping the student to give a clear explanation.

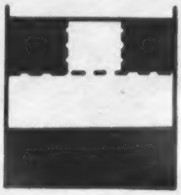


First Second

S: Take one scoop from the first carton and put it into the second carton.



First Second



Second

S: Then the probability of red is $\frac{5}{9}$ and of white is $\frac{4}{9}$.

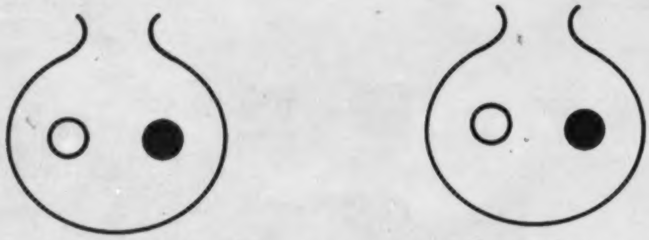
Again, some student may suggest using a probability tree - see note at the end of this lesson. You should accept this explanation and present it. It would be good to look at both ways of solving the problem.

Exercise 4

T: So Allen also suggested an unfair game; of course, this time it is in his favor because red has the larger probability. Do you think we can find a way to change this game so that it will be fair?

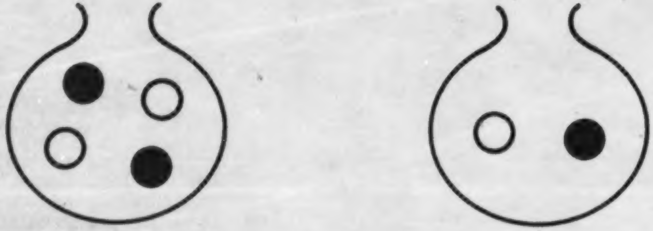
Let the students discuss some possibilities. If someone suggests putting all the marbles in one bag (three of each color) and just drawing one, accept the change as okay but say that we would like to keep the two bags and use a similar experiment (take a marble from the first, put it into the second, and then take a marble from the second). The only change we wish to make is what marbles are in the bags.

Give the students a few minutes to consider various possibilities. Probably you will get a suggestion to put the same number of each color marble into each bag. For example,



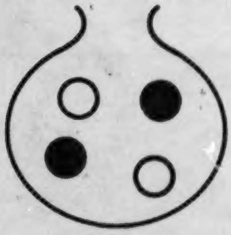
First Bag Second Bag

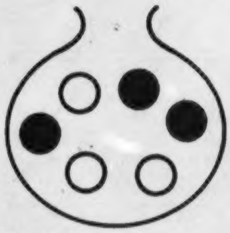
or



First Bag Second Bag

or





Of course, here more than six marbles are being used.

and so on.

If some such suggestion is not made, you should make it yourself. However, be sure to give the students time to examine some possibilities.

T: Now, I believe the game is fair. Can someone convince us that this is a fair game?

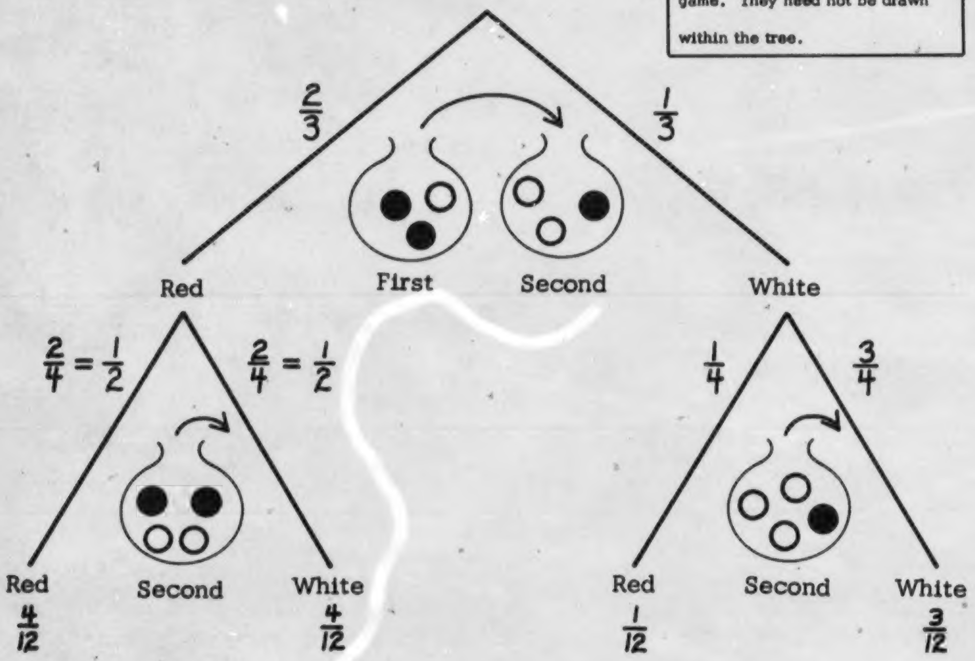
S: There is no difference between the white and red marbles. You could change all the reds to white and whites to red; it is the same. They must then both have the same probability.

It is likely that this form of reasoning will not be suggested immediately. If it is not, explain it yourself. Perhaps, for the benefit of some of your students you may wish to go through some form of analysis (similar to that used before) suggested by the students.

Note on Exercise 2

A student may suggest using a probability tree to help find the probabilities for red and white.

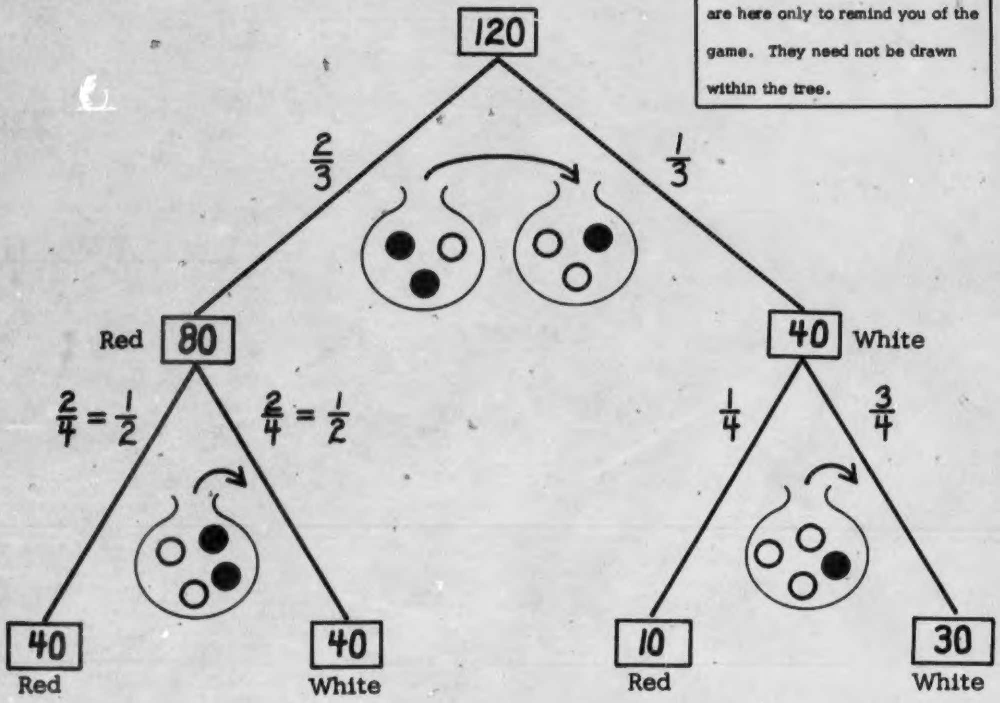
The pictures of bags of marbles are here only to remind you of the game. They need not be drawn within the tree.



2/3
1/3
Red
First
Second
White
2/4 = 1/2
2/4 = 1/2
1/4
3/4
Red
4/12
Second
White
4/12
Red
1/12
Second
White
3/12

This shows that the probability for red is $\frac{5}{12}\left(\frac{4}{12}+\frac{1}{12}\right)$ and for white is $\frac{7}{12}\left(\frac{4}{12}+\frac{3}{12}\right)$. To find these probabilities, suggest a discussion (similar to the dialogue of Section 1.3) of what to expect if the game is played say 120 times.

The pictures of bags of marbles are here only to remind you of the game. They need not be drawn within the tree.



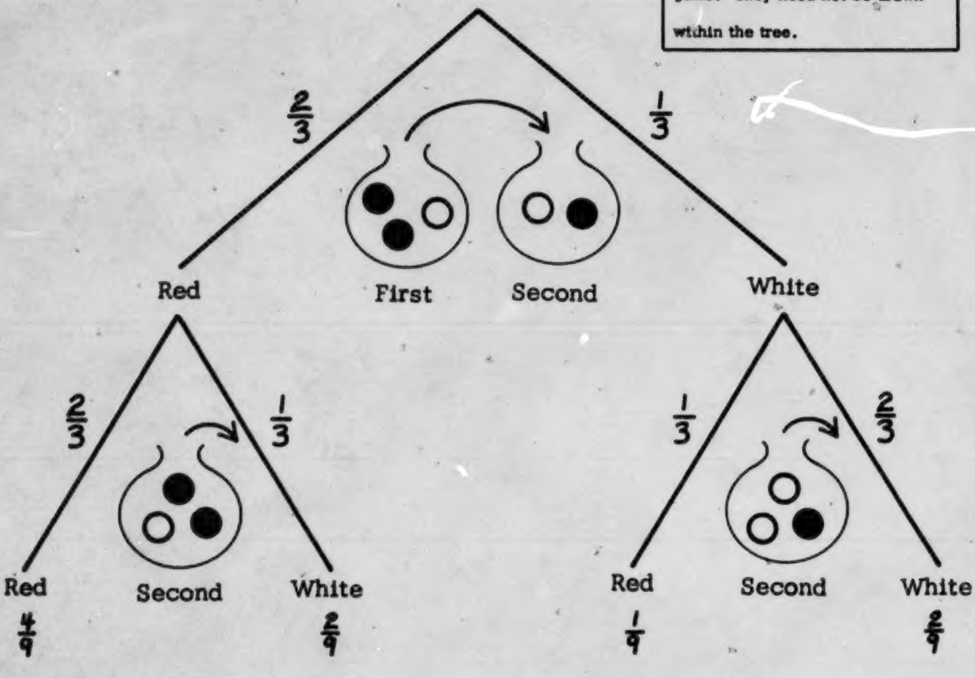
This game is not fair and favors white.

If this suggestion is made by a student, accept it as a good way to look at the problem and discuss it with the class. Then return to suggest analysis of the problem as described in the lesson. The advantages of using the proportion ideas in analyzing the problem are that they give an alternative to using a probability tree and operating with fractions, while at the same time providing experience with important mathematical notions of proportionality.

Note on Exercise 3

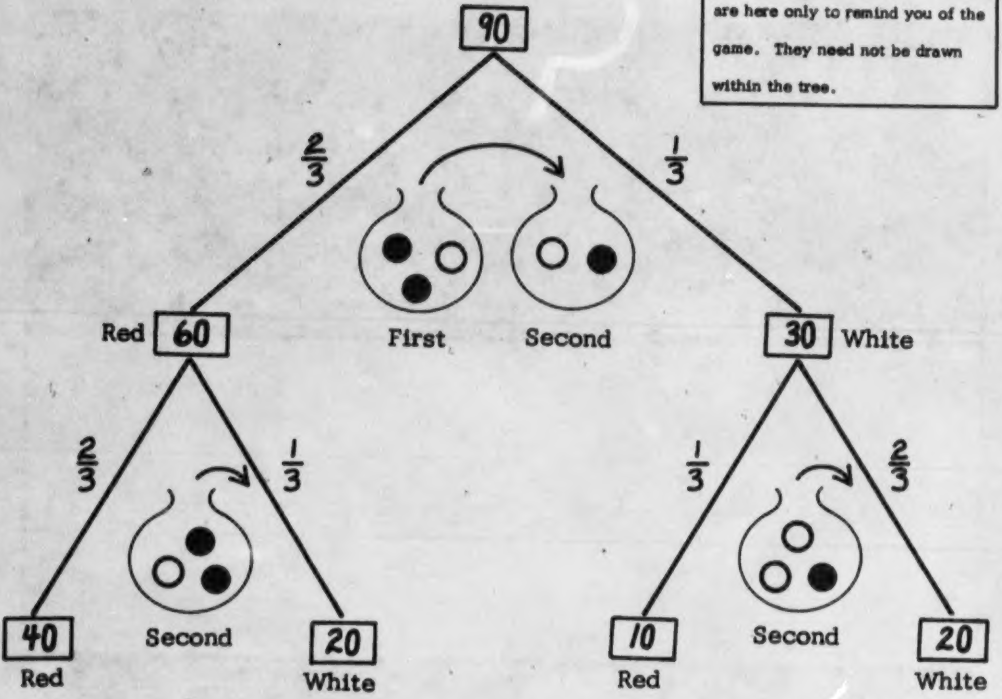
A student may suggest using a probability tree.

The pictures of bags of marbles are here only to remind you of the game. They need not be drawn within the tree.


2/3
1/3
First
Second
Red
White
2/3
1/3
Second
Red
4/9
White
2/9
1/3
2/3
Second
Red
1/9
White
2/9

This tree shows that the probability for red is $\frac{5}{9}\left(\frac{4}{9} + \frac{1}{9}\right)$ and for white is $\frac{4}{9}\left(\frac{2}{9} + \frac{2}{9}\right)$. To find these probabilities, suggest a discussion (similar to the dialogue of Section 1.3) of what to expect if the game is played say 90 or 180 times.

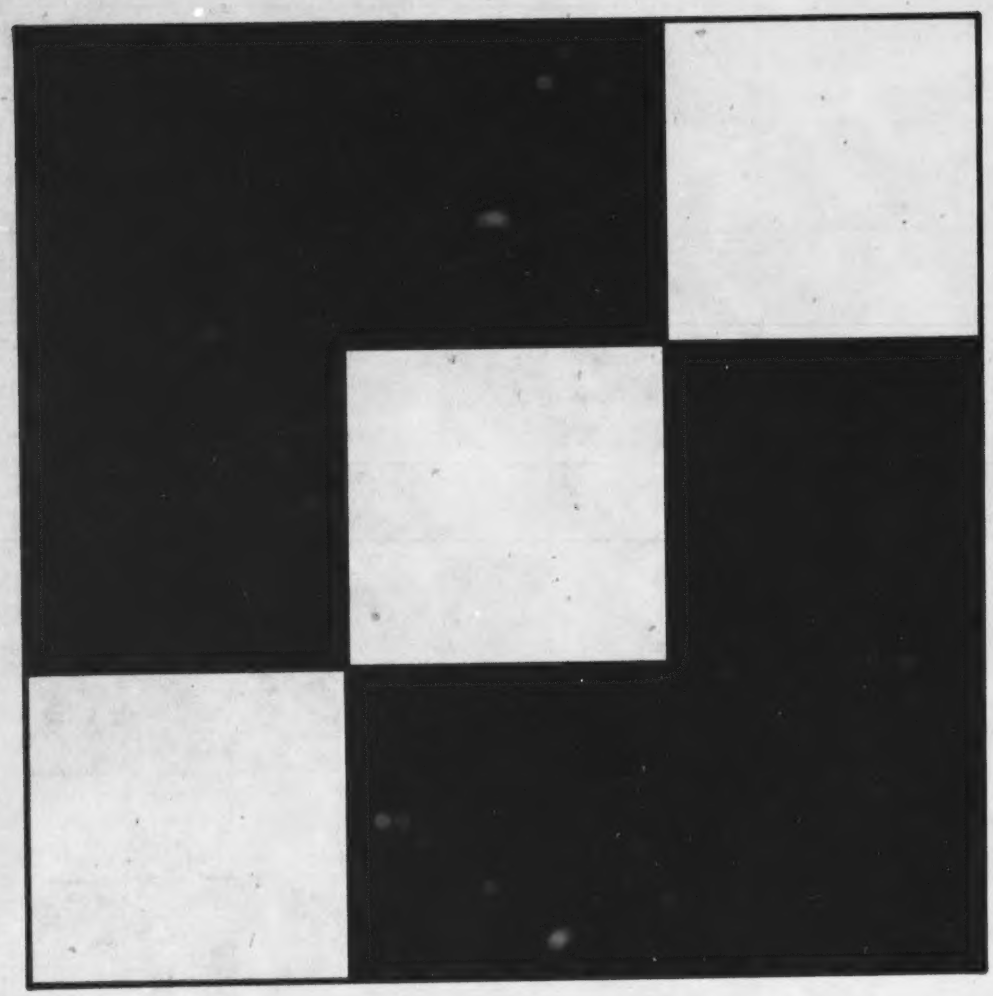
The pictures of bags of marbles are here only to remind you of the game. They need not be drawn within the tree.


90
2/3
1/3
Red
60
First
Second
30
White
2/3
1/3
1/3
2/3
Second
Second
40
Red
20
White
10
Red
20
White

The game is not fair and favors red.

5P

5P



5P